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Tl.0. THE HOMOGENEOUS TRANSMISSION LINE

The electrical interconnections in digital systems are systems of transmission lines rather than just wires. In some applications it is possible to view the transmission lines as simple conductors. In the main, however, we must either plan the interconnection system as transmission lines or keep the transmission line viewpoint in mind when designing or analyzing such interconnections. For the most part the signals passing over the interconnecting paths are pulses rather than sine waves. Consequently pulse excitation of transmission lines is the main topic of concern in this chapter.

The general transmission line considered here is a homogeneous assemblage of at least two conductors. The geometric and physical parameters of the line are assumed to be constant everywhere along the line. The line is characterized by a number of parameters:

- a. The capacitance C is the capacitance per unit length between the pair of conductors constituting the line.
- b. The inductance L is the inductance per unit length of line. The measured value of L may include mutual inductance as well as self-inductance.
- c. The resistance R per unit length is the resistance of the conductors to the passage of current.
- d. The conductance G is the leakage conductance per unit length due to losses in the dielectric separating the two conductors.

An infinitesimal element of transmission line is shown in Fig. Tl. Two fundamental equations describing line behavior can be obtained from this elementary circuit element. As $\Delta x \rightarrow 0$

$$\frac{\partial v}{\partial x} = -iR - L \frac{\partial i}{\partial t} \quad (T-1)$$

$$\text{and } \frac{\partial i}{\partial x} = -Gv - \frac{C\partial v}{\partial t} \quad (T-2)$$

These two equations combine to give the telegrapher's equation.

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2} + (GL+RC) \frac{\partial v}{\partial t} + GRv \quad (T-3)$$

This relation can be integrated in the special cases of sinusoidal excitation and in the transient case where Laplace transforms afford a solution.

Of particular interest is the case of the line that is lossless or for practical purposes is treated as having no losses. For the lossless line $G \cong 0$ and $R \cong 0$ and Eq. (T-3) reduces to

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2} \quad (T-4)$$

We can define

$$u = 1/\sqrt{LC} = 1/\delta \quad (T-5)$$

where u is the propagation velocity and δ is the propagation constant or delay per unit length. Then Eq. (T-4) takes the form

$$\frac{\partial^2 v}{\partial x^2} - \delta^2 \frac{\partial^2 v}{\partial t^2} = 0 \quad (T-6)$$

The general solution to the differential equation of Eq. (T-6) is of the form

$$v(x,t) = f(x-ut) + g(x+ut) \quad (T-7)$$

where $f(x,t)$ and $g(x,t)$ are arbitrary functions. For the lossless line equations (T-1) and (T-2) become

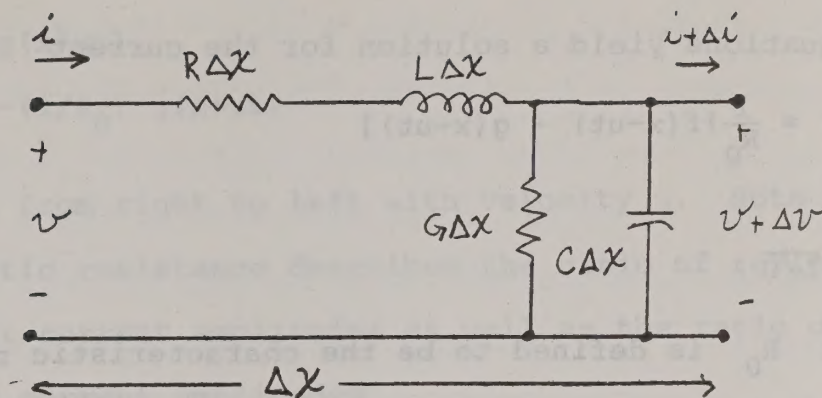


Fig. T-1. Elemental length of transmission line

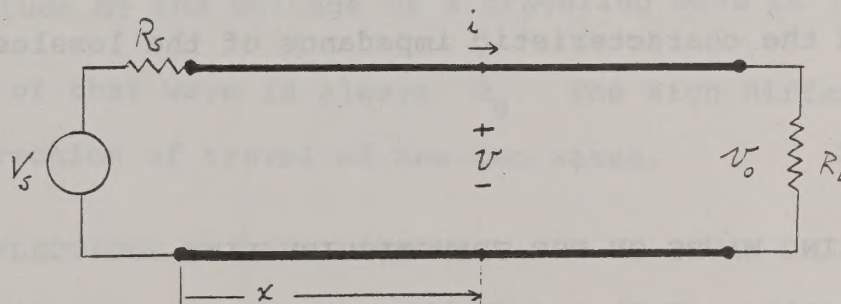


Fig. T-2. Example line defining traveling wave

$$\frac{\partial v}{\partial x} = -L \frac{\partial i}{\partial t} \quad (T-8)$$

$$\frac{\partial i}{\partial x} = -C \frac{\partial v}{\partial t} \quad (T-9)$$

These two equations yield a solution for the current

$$i(x,t) = \frac{1}{R_0} [f(x-ut) - g(x+ut)] \quad (T-10)$$

where

$$R_0 = \sqrt{L/C} \quad (T.11)$$

The constant R_0 is defined to be the characteristic resistance of the line. In the case of the lossy line the equivalent resistance is the characteristic impedance, Z_0 .

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad (T.12)$$

The delay and the characteristic impedance of the lossless line are related.

$$\delta = R_0 C \quad (T-13)$$

T1.1. TRAVELING WAVES ON THE TRANSMISSION LINE

The relations (T-7) and (T-10) are

$$v(x,t) = f(x-ut) + g(x+ut)$$

$$i(x,t) = \frac{1}{R_0} [f(x-ut) - g(x+ut)]$$

These relations can be viewed as describing a pair of waves on the transmission line traveling with the same speed but in opposite directions. For example,

$$v_i = f(x-ut) \quad (T-14)$$

$$i_i = v_i/R_0 = (1/R_0) f(x-ut) \quad (T-15)$$

describe the (incident) wave traveling with constant velocity, u , from left to right in Fig. T-2. Another wave (reflected) is described by

$$v_r = g(x+ut) \quad (T-16)$$

$$i_r = -(1/R_0) g(x+ut) \quad (T-17)$$

and travels from right to left with velocity u . Note that the characteristic resistance describes the ratio of reflected voltage and incident current amplitudes as well as the ratio of incident voltage and current amplitudes

$$\frac{v_i}{i_i} = R_0 ; \quad \frac{v_r}{i_r} = -R_0 \quad (T-18)$$

That is, at any point on the line the magnitude of the ratio of the amplitude of the voltage of a traveling wave to the current amplitude of that wave is always R_0 . The sign difference is due to the direction of travel of the two waves.

T1.2. REFLECTIONS AT TERMINATIONS

Consider the load end of the line as shown in Fig. T-3. At the point of termination there will be some voltage, v , across the terminals and some current i through the load. The load current and voltage must be

$$v = v_i + v_r \quad (T-19)$$

$$\text{and } i = i_i - i_r \quad (T-20)$$

However, $v = iZ_L$ as well and so that

$$\frac{v_i + v_r}{Z_L} = \frac{v_i}{R_0} - \frac{v_r}{R_0} \quad (T-21)$$

Solving for the reflected signal,

$$v_r \left(\frac{1}{Z_L} + \frac{1}{R_0} \right) = v_i \left(\frac{1}{R_0} - \frac{1}{Z_L} \right) \quad (\text{T-22})$$

$$v_r = v_i \left(\frac{Z_L - R_0}{Z_L + R_0} \right) \quad (\text{T-23})$$

Here the quantity

$$\rho = \left(\frac{Z_L - R_0}{Z_L + R_0} \right) \quad (\text{T-24})$$

is called the reflection coefficient since

$$v_r = \rho v_i \quad (\text{T-25})$$

The viewpoint expressed is that an incident voltage signal arriving at the load termination produces a reflected voltage as given by Eq. (T-25). The reflected current can also be found

$$i_r = -\rho i_i \quad (\text{T-26})$$

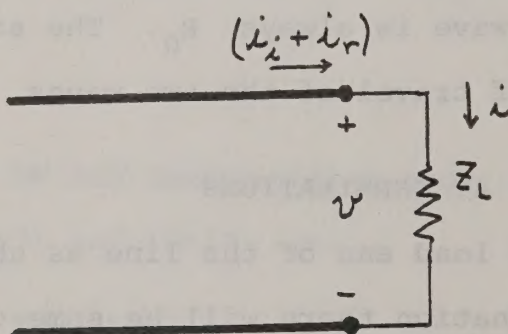


Fig. T-3. Load end of the line at $t = \lambda \delta$

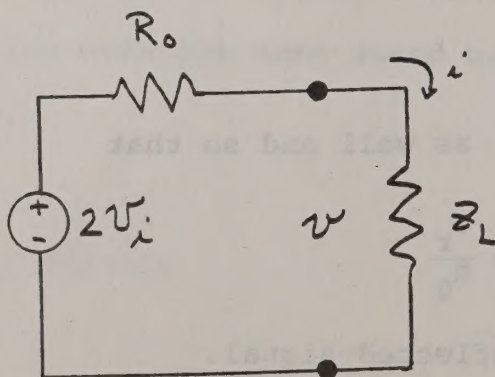


Fig. T-4. Equivalent circuit of the load end of the line at $t = \lambda \delta$

If the terminating element is a resistor $R_L = R_0$, then $\rho = 0$ and there will be no reflected voltage or current. That is, an incident wave reaching this "matched" termination will stop there. An incident wave arriving at a mismatched termination will produce a reflected wave. Obviously any terminating or line voltage will be the result of a sequence of traveling waves each producing a reflected wave which in turn produces a reflected wave, etc. It can be shown that the voltage or current at any point on the line (including terminations) at time t can be found from the superposition of all of the voltage or current waves appearing at that point at or prior to time t . Each change in line voltage or current is due to the passage of an incident and/or reflected signal past that point.

Consider the arrival of a wave at the termination of Fig. T-3: At the time of arrival of the incident wave v_0 a change in voltage can occur.

$$v_0 = v_i + v_r \quad (T-27)$$

$$= v_i(1+\rho) \quad (T-28)$$

This relation can be put in a form that suggest an equivalent circuit.

$$v_0 = v_i \left(1 + \frac{Z_L - R_0}{Z_L + R_0} \right) \quad (T-29)$$

$$v_0 = 2v_i \left(\frac{Z_L}{Z_L + R_0} \right) \quad (T-30)$$

This last form for the output voltage due to the arrival of an incident signal is very revealing. It describes the equivalent circuit of Fig. T-4. That is, the voltage produced by the incident

voltage v_i can be found by considering the line to be a signal voltage source of amplitude $2v_i$ with a source resistance R_0 . This equivalent circuit is very useful.

T1.3. THE INPUT EQUIVALENT CIRCUIT

At a line input a different condition exists. An input signal source can cause the line voltage to change. This change launches a wave into the transmission line from that point. This is an initial incident wave and no reflection is said to occur in the sense previously used. Let us find the equivalent circuit of the line as seen from the signal source. In addition let the line carry an initial current, I_{L0} , and be charged to an initial voltage, V_{L0} . At the instant $t = 0$ when a change v_s occurs as indicated in Fig. T-5

$$\Delta v = V_{L0} - V_x \quad (T-31)$$

The change in input voltage, Δv , is forced upon the line and causes a change in line current as well. This new wave launched on the line will be an incident wave arriving at points to the right of the input terminals at a time $t > 0$. Certainly the relation of Eq. (T-18) holds and

$$\Delta i = \Delta v / R_0 \quad (T-32)$$

Thus the impedance of the equivalent circuit representing the line is Z_0 . To find the rest of the equivalent circuit we can ask what the open circuit line voltage would be at $t = 0^+$? This we can do by adjusting the voltage V_x to that value which will produce $i = 0$. That is, V_x will be adjusted until $i = 0$ when the switch is made at $t = 0^+$. Then the corresponding value of V_x is the open circuit voltage representing the line at $t = 0^+$.

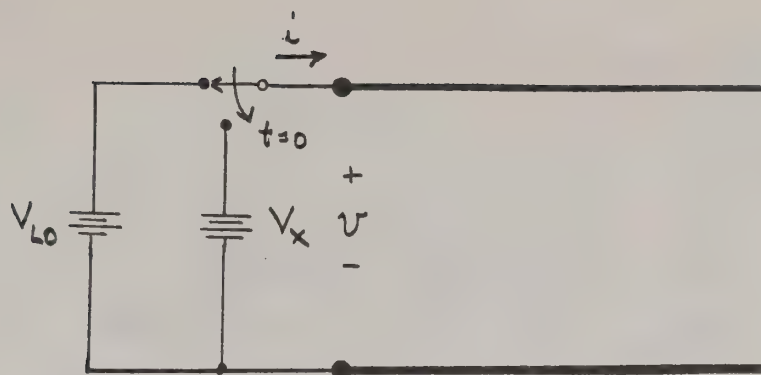


Fig. T-5. Solving for the initial equivalent circuit

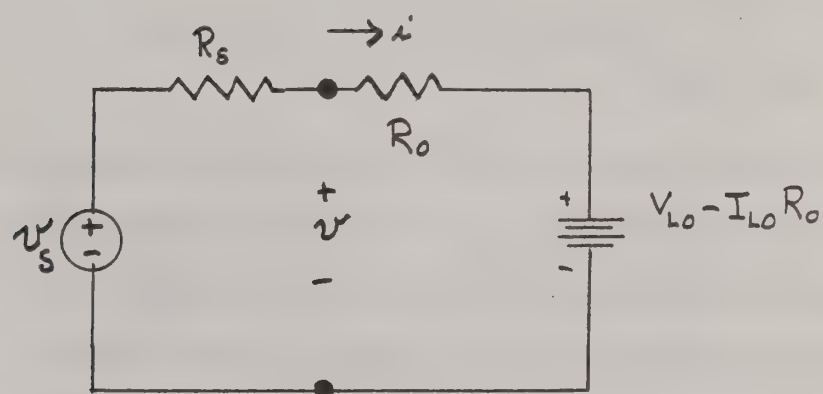


Fig. T-6a. Thevenin input equivalent circuit at $t = 0^+$

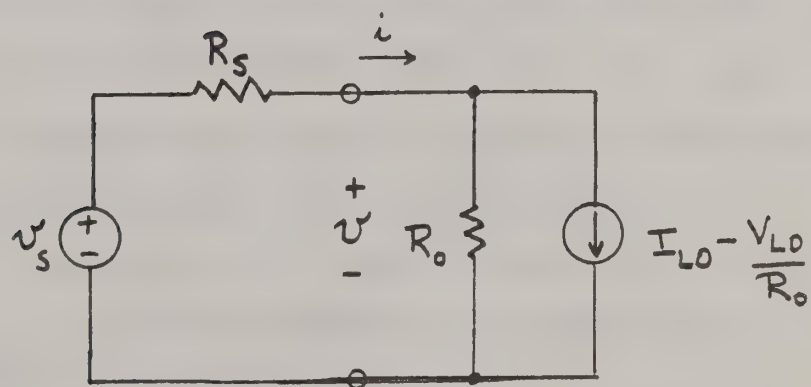


Fig. T-6b. Norton equivalent of line at $t = 0^+$

At $t = 0^+$, then,

$$\Delta i = -I_{L0} \quad (T-33)$$

$$\Delta v = V_x - V_{L0} \quad (T-34)$$

Since $\Delta v / \Delta i = R_0$,

$$\frac{V_x - V_{L0}}{-I_{L0}} = R_0 \quad (T-35)$$

Giving the solution

$$V_x = V_{L0} - I_{L0} R_0 \quad (T-36)$$

That is, the equivalent circuit representing the line seen by the generator is an impedance R_0 to any transient signals produced by the source in series with a constant voltage $V_{L0} - I_{L0} R_0$ representing the line initial conditions as indicated in Fig. T-6a.

Alternatively the Norton equivalent circuit could have been found by applying a current I_x such that when the switch is made at $t = 0^+$, the line input voltage becomes $v = 0$. The Norton equivalent input circuit for the line is shown in Fig. 6.6.

Let us apply this equivalent circuit to a problem. For example, consider the circuit of Fig. T-7 where the transistor has been saturated for a long time and at $t = 0$ becomes cut off. Initially $v \cong 2.5V$ and $i = -5V/100\Omega \cong -50mA$. The line may be replaced by a resistor R_0 in series with a voltage source

$$V_{L0} - I_{L0} R_0 = 2.5 - (-0.05)(200) = +12.5V.$$

The equivalent circuit for this example is shown in Fig. T-8a, and the resultant $v(+)$ in Fig. T-8b.

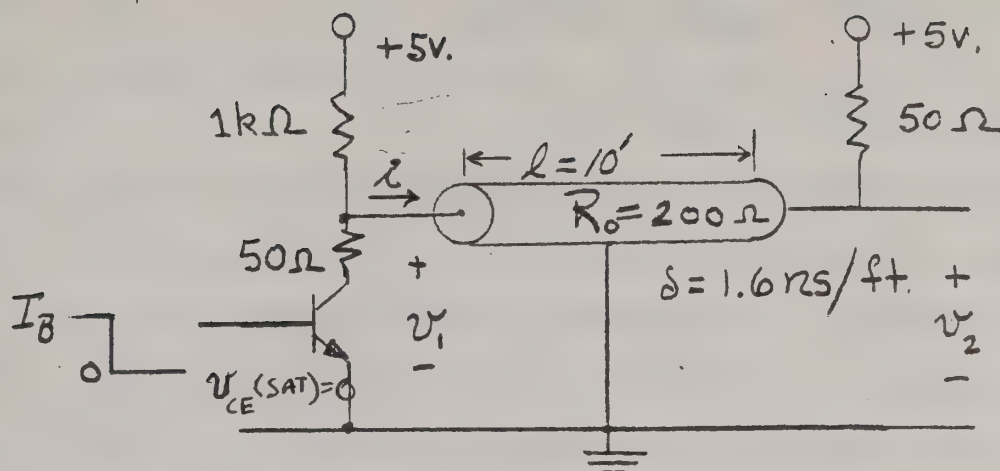
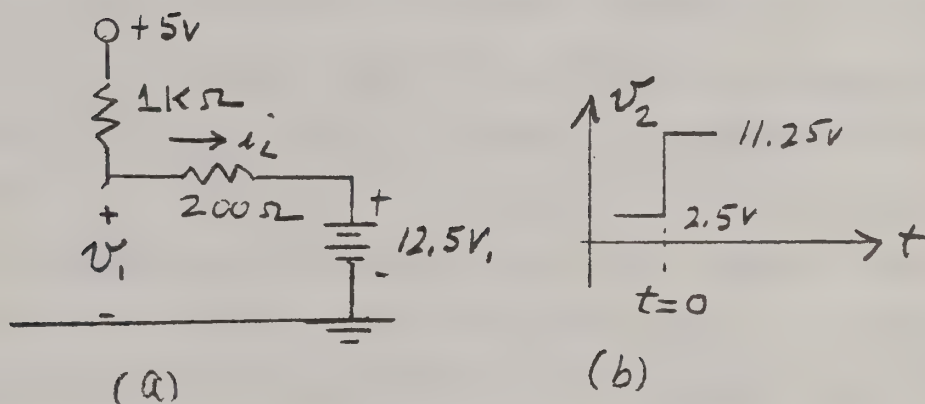


Fig. T-7. Example problem

Fig. T-8 (a) Equivalent input circuit (b) waveform of input signal near $t = 0$

Let us continue this example. The jump in input voltage and input current constitute the incident waveform

$$v_i = \Delta v = 11.25 - 2.5 = 8.75 \text{ V}$$

$$i_i = \frac{8.75 \text{ V}}{200 \Omega} = 43.75 \text{ mA}$$

The latter value could also be obtained from the equivalent circuit of Fig. T-8a:

$$i_i = \frac{5-12.5V}{1.2k\Omega} - (-50mA) = 43.75 \text{ mA}$$

T1.4. TRANSMISSION AND REFLECTION

This wave travels from left to right towards the load, R_L . The incident voltage is described by $v_i = f(x-ut)$ where f is an arbitrary function. It is obvious in this example that the arbitrary function is a sudden 8.75 V increase in voltage, a step function. The velocity of propagation is given as 1.6ns/ft. Since the line is 10 feet long, this incident step function will arrive at the load at $t = 16 \text{ ns}$.

The line is lossless and there is no attenuation of the signal. Hence an incident 8.75 V step of voltage appears at the end of the line at $t = 16 \text{ ns}$. There are a number of ways to determine what happens at the instant that the incident wave arrives at the load. Let us examine that instant using the equivalent circuit of Fig. T-4 derived earlier. The equivalent circuit pertinent to this example at $t = 16 \text{ ns}$ is shown in Fig. T-9. The line is represented by an equivalent circuit containing the battery representing the initial

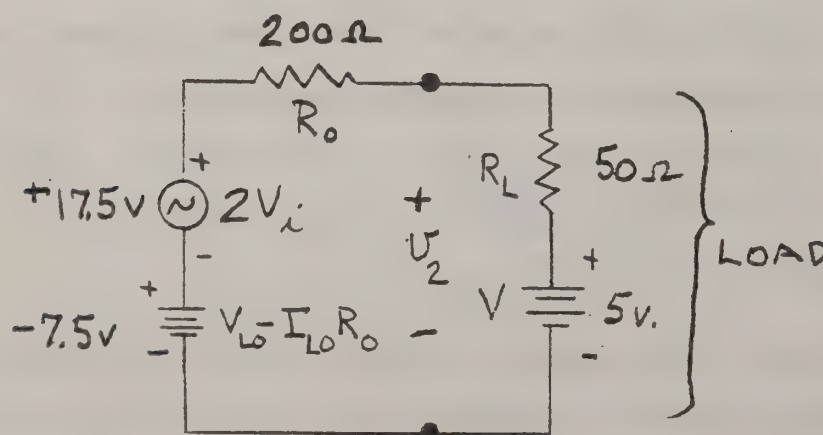


Fig. T-9. The right-hand end of the line at the time $t = 16 \text{ ns}$

conditions, $V_{L0} = I_{L0}R_0$, in series with a generator $2v_i$, due to the incident voltage, and R_0 , the line characteristic resistance.

Since the initial current is into the line from the load, the initial current, I_{L0} , is a positive 50 mA in contrast to the other end of the line where $I_{L0} = -50$ mA. Consequently, the generator representing the initial conditions is of necessity different. Solution of this circuit provides the information that the load end voltage jumps suddenly from $V_{L0} = 2.5$ V. to $v_0 = 6$ V. as the incident wave arrives. The jump in load end voltage is $\Delta v_0 = 6.0 - 2.5 = 3.5$ V.

The jump in load voltage can be obtained by simpler computations. For example, the reflection coefficient is

$$\rho_2 = \frac{50 - 200}{50 + 200} = -0.6$$

Knowing this and that the incident voltage is 8.75V, we can calculate that the voltage component of the reflected wave to be

$$V_r = \rho_2 V_i = (-0.6)(8.75) = -0.525 \text{ V.}$$

The voltage at the termination at $t = 16$ ns jumps from the initial value of 2.5 volts to

$$\begin{aligned} v_2 &= V_{L0} + v_i + v_i \\ &= 2.5 + 8.75 - 5.25 = 6 \text{ V.} \end{aligned}$$

This second calculation using the reflection coefficient is a simpler calculation than the first using the equivalent circuit. We will use the equivalent circuit approach for conceptual purposes,

for discussion of graphical solution techniques, and wherever either the termination or the source impedance changes as in the example at $t = 0$. For the majority of the cases, however, the reflection coefficient calculation proves to be far simpler.

T1.5. THE REFLECTION DIAGRAM

A formal calculation method has been evolved which systematizes the calculations involved in the case of multiple reflections. Where the line is not matched at either end numerous reflections may have to be calculated before the traveling wave amplitude has fallen to the point where it is of no further interest. The reflection diagram, or lattice diagram, and variations of it have proved to be of great value in formalizing and organizing such calculations.

The reflection diagram represents distance in the horizontal plane (x-axis) and time in the vertical plane. The traveling waves are represented by diagonal lines from left to right or vice versa. A reflection diagram for the preceding example is shown in Fig. T-10.

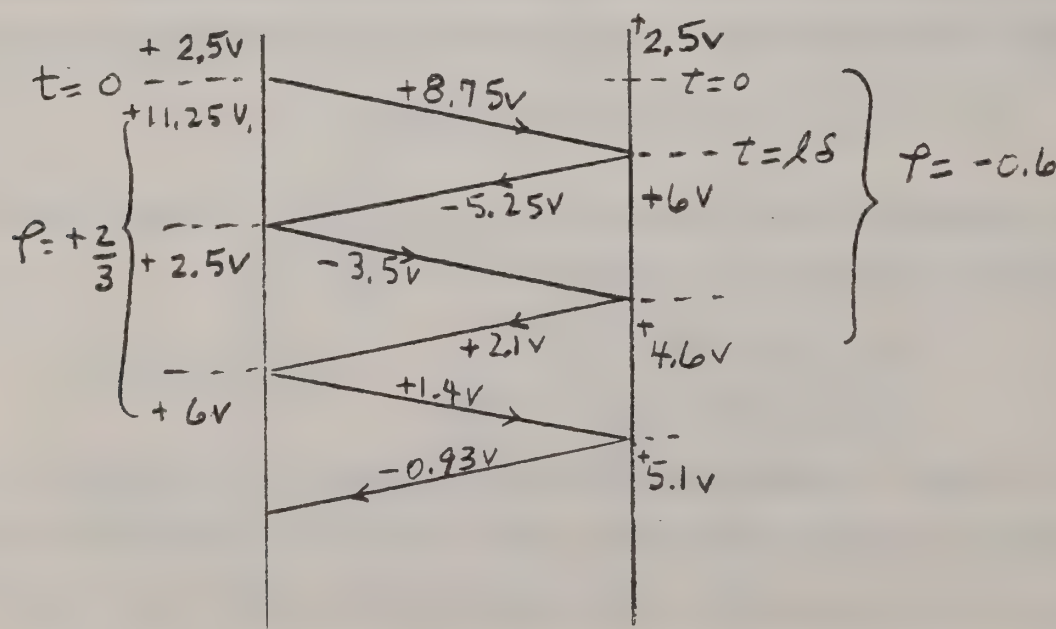


Fig. T-10. Reflection diagram for example problem of Fig. T-7.

Note that the voltage at each end of the line prior to $t = 0$ is 2.5 V. At $t = 0^+$ the input voltage jumps to 11.25 V as was calculated previously, and a +8.75 V transient increase in line voltage is produced. The traveling (incident) wave moves from source (left) to the load (right) with a velocity $u = 1/\delta$. At time $t = \delta = 16$ ns the incident signal reaches the load and the load voltage jumps to 6 V. as was previously found. Since initially +2.5 V existed at the load, the incident voltage signal is +8.75 V, and the resultant voltage is 6 V. Therefore

$$6v = V_{L0} + v_i + v_r$$

$$6v = 2.5 + 8.75 + v_r$$

$$\text{and } v_r = -5.25 \text{ V}$$

This could also have been found from

$$\begin{aligned} v_r &= \rho_2 v_i \\ &= (-0.6)(8.75\text{V}) = -5.25\text{V} \end{aligned}$$

where

$$\rho_2 = (R_L - R_0) / (R_L + R_0)$$

From this point on the cycle repeats itself. At $t = 2\delta = 32$ ns the -5.25 V. signal (reflected) from the right hand edge arrives at the left hand end of the line. There it encounters the 1000 Ω termination provided by the collector load resistance since the transistor is cut off and appears as an open circuit. Instead of drawing the equivalent circuit with initial condition and solving for the new voltage at $t = 2\delta^+$, we may solve the transient problem alone. A -5.25 V traveling wave going right to left reaches the input and produces a reflection $v_r = \rho_1 v_{inc}$ where $\rho_1 = \frac{1000-200}{1000+200} = +2/3$. This is indicated on the reflection diagram of Fig. T-10 by the -3.5 V signal traveling from left to right.

Since the arriving signal has a -5.25 V amplitude and the reflected signal a -3.5 V amplitude, the voltage at the left hand end of the line shows a -8.75 V change in amplitude at $t = 2\ell\delta^+$.

The traveling wave bounces back and forth between the line ends until its amplitude is so small that it is of no interest. The waveform of $v(t)$ can easily be drawn from the reflection diagram since both amplitude and time information are available. The resulting waveforms at both ends of the line are shown in Fig. T-11.

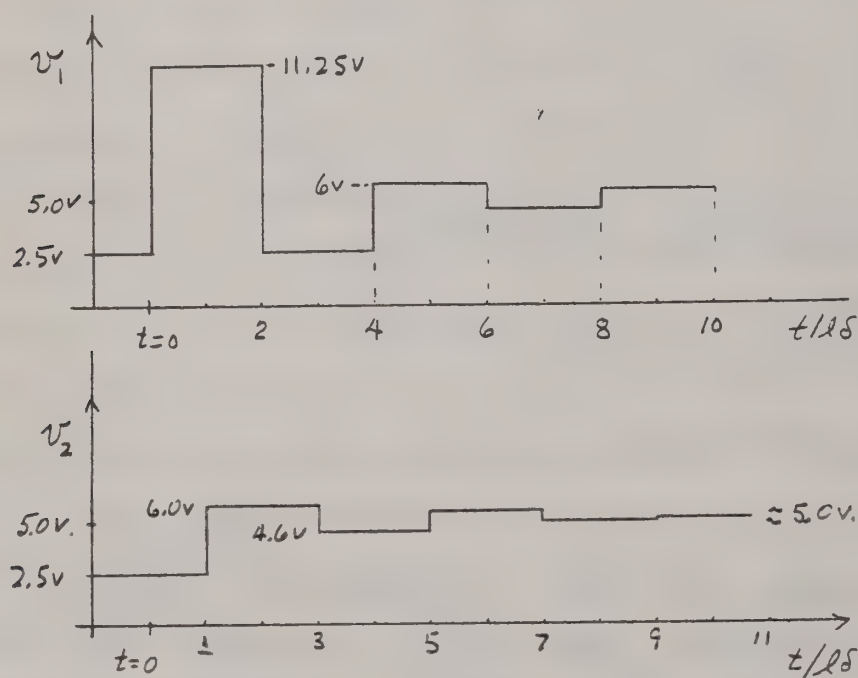


Fig. T-11. Wave eforms at both ends of the line for the example of Fig. T-7.

In this case each transition is a step since the original transient was a step and all terminations are resistive.

T1.5. TERMINATIONS AND THEIR EFFECTS

Considerable information is present in the waveform of the

response of a transmission line to an initial step of voltage. For example, if there were no reflections returning to the signal source we know that the far end of the line may be terminated in a resistance R_0 . We have to hedge a bit since there are non-linear terminations which will produce no reflections. If the termination is a simple L, R, or C, we can determine the nature of that termination from observing the input waveform in response to an applied step of voltage.

Let us examine a number of the most frequently encountered terminations. Assume the line to be terminated at the right hand end in a resistance, R_L and driven on the left hand end by a step of voltage through a resistance R_S as indicated in Fig. T-12.

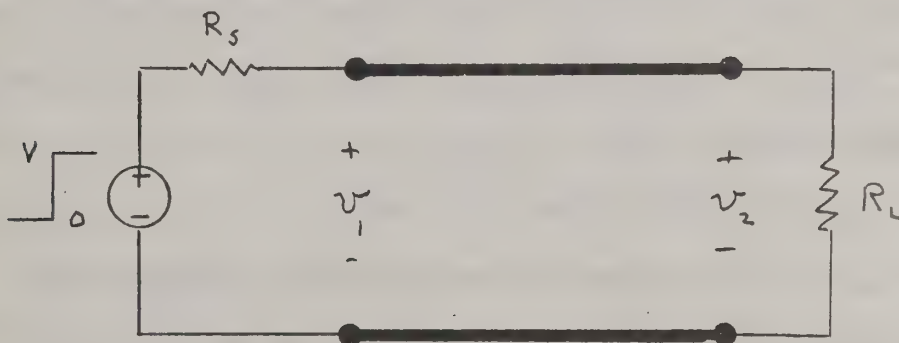


Fig. T-12. Line termination

MATCHED TERMINATIONS

Whenever $R_L = R_0$, the line is said to be matched at the load end. Since $R_L = R_0$, $\rho_2 = 0$ and there are no reflections. At a time $t = \ell\delta$ the transient has propagated to the end of the line as indicated in Fig. T-13a.

The line may also be matched at the sending end. For example, if $R_S = R_0$ and $R_L \neq R_0$, the reflections indicated in Fig. T.13b occur. In this case the line is said to be back matched or series

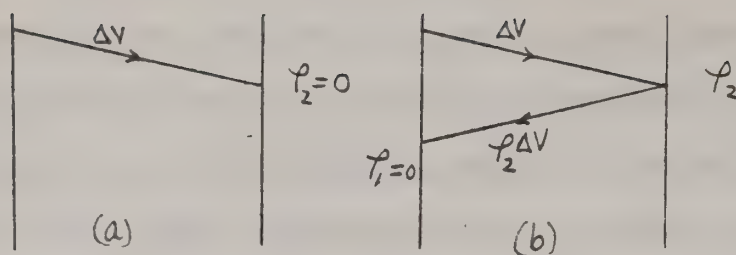


Fig. T-13. Matched transmission lines

(a) matched at load

(b) back matched (at the source)

matched. The transient dies out after a complete round trip $t = 2\ell\delta$. The source end of the line sees one reflection while the load end does not.

MISMATCHED TERMINATIONS

Case A: $R_S > R_0$, $R_L > R_0$

In this case both reflection coefficients are positive. At either end of the line each incident signal and each reflection add to the voltage at that end of the line. The resultant reflection diagram is shown in Fig. T-14a and from it the waveform of Fig. T-15a will be found.

Case B: $R_S > R_0$, $R_L < R_0$

The initial input signal change is positive, but the signal reaching the right hand end of the line is alternately positive and negative as indicated in Fig. T-14b. The waveform of the input signal is shown in Fig. T-15b and has an easily recognizable ringing shape.

Case C: $R_S < R_0$, $R_L > R_0$

In this case the initial input is positive and so is the next change in the line input voltage. There is a ringing effect as in the previous case, but the change in voltage at $t = 2\ell\delta$ easily distinguishes the two.

Case D: $R_S < R_0$, $R_L < R_0$

In this case each reflection is of opposite sign to the incident signal causing it. The resulting waveform, shown in Fig. T-15d, has a considerable overshoot.

The response in each of the cases discussed above is sufficiently distinctive to make the step response of a transmission line a useful analysis tool. In fact a technique called Time Domain Reflectometry (TDR) makes use of a pulse generator with a very short rise time and a high speed oscilloscope. TDR measurements are used to find and analyze faults, imperfections and the properties of terminations of transmission lines.

Tl.6. MISMATCHED LINES

Reflections can arise from the change in transmission line impedance that occurs at a junction of two different transmission lines, a sharp bend, kink or break in the shield of a line, or at a connector between lines or line and circuit board, etc. Any transition between one line of characteristic resistance Z_A and another of characteristic resistance Z_B can result in a reflection. As a practical matter changes in characteristic resistance of 5 to 10% can usually be ignored, but beyond that some analysis of their effect may be needed.

Consider the application of Fig. T-16 where a signal v_s passes through two different transmission lines to arrive at the load.

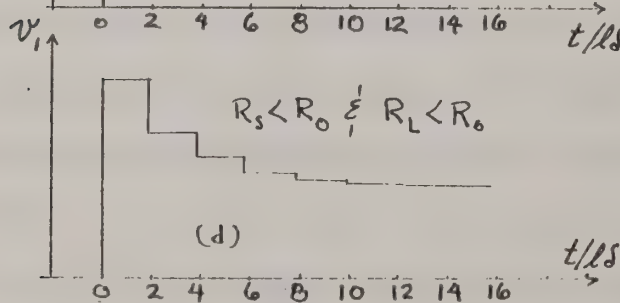
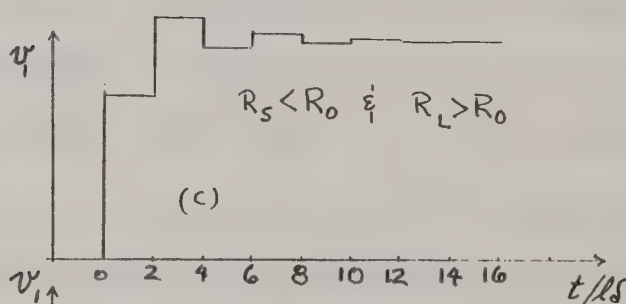
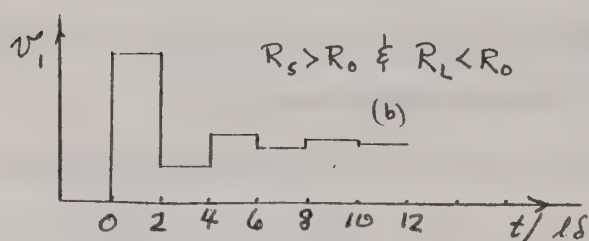
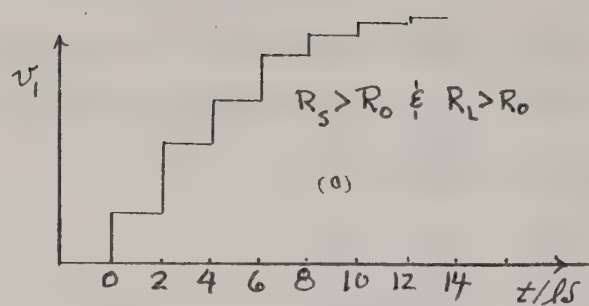


Fig. T-15. Waveforms for mismatched lines

$$v_2 = \frac{2v_{inc}R_{0B}}{R_{0B}+R_{0A}} \quad (T-37)$$

Line B acts as a load on line A. The voltage at that junction could also be expressed as

$$v_2 = v_{inc}(1 + \rho_{AB}) \quad (T-38)$$

$$\text{where } \rho_{AB} = \frac{R_{0B}-R_{0A}}{R_{0B}+R_{0A}} \quad (T-39)$$

The reflection coefficient ρ_{AB} describes the reflection of a signal passing from line A into line B as diagramed in Fig. T-17b. Since the voltage at the junction of the two lines is the sum of the incident and reflected waves, the wave launched into line B is

$$(v_{inc})_B = (v_{inc})_A (1+\rho_{AB}) \quad (T-40)$$

This causes a complication not encountered before. There are now two waves traveling on the transmission lines and two waves to keep track of on the reflection diagram. However, if care is exercised this can be accomplished accurately.

It should be noted that there are two different reflection coefficients at the junction of two lines. For signals passing from line A into line B we have ρ_{AB} as defined in Eq. (T-39). For signals passing from line B into line A we have

$$\rho_{BA} = \frac{R_{0A}-R_{0B}}{R_{0A}+R_{0B}} \quad (T-41)$$

$$\text{and } \rho_{BA} = -\rho_{AB} \quad (T-42)$$

EXAMPLE T1

As an example consider that line A is an RG58AU cable 12 feet long and has $R_{0A} = 52 \Omega$ and $\delta_A = 1.48 \text{ ns/ft.}$ Line B has

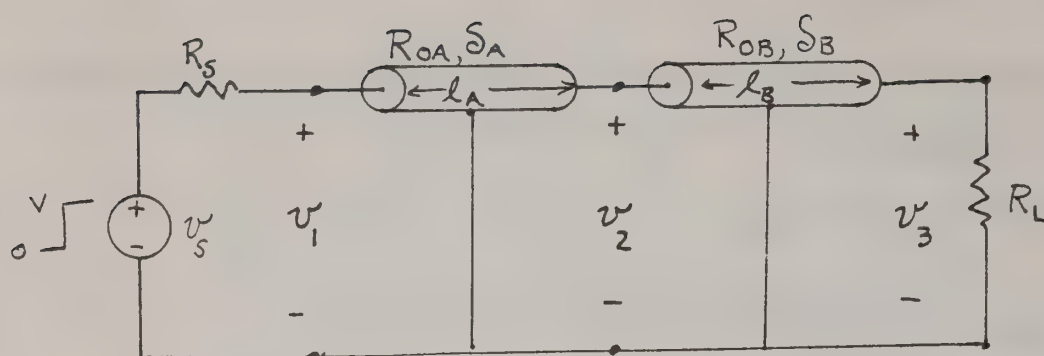


Fig. T-16. Two transmission lines with different characteristics are used to carry the signal

At the generator end the initial conditions, at $t = 0^+$, are established by the source and transmission line A. After the incident wave has traveled the length of line A ($t = l_A \delta_A$) it reaches the junction of line A and line B. Each line at that time can be represented by its equivalent circuit as indicated in Fig. T-17. The initial conditions are assumed to be that $I_{L0} = 0$, $V_{L0} = 0$.

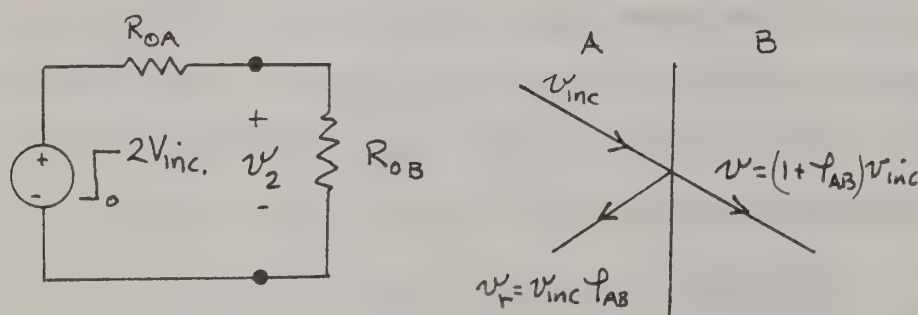


Fig. T17. (a) Equivalent circuit at the juncture of line A and line B at $t = l_A \delta_A$ (b) Incident and reflected signals.

From the equivalent circuit of Fig. T-17a the line voltage v_2 can be found as

From the equivalent input circuit at $t = 0$ the incident wave is found to be $+3.42$ V. This reaches the line junction at 17.8 ns and produces a 1.35 V reflection and a $1.35 + 3.42 = 4.77$ V signal continuing on in line B. As indicated in Fig. T-19 this signal finally arrives at the right hand end of line B at $t = 56.6$ ns and it too causes a reflection as the line voltage jumps to $+6.45$ V. Meanwhile the original signal in line A was reflected first from line B, then from the source end, then from the junction of the two lines again. Note that there are already instances when it is necessary to keep track of 3 or 4 signals which are simultaneous in time. However, once a signal magnitude becomes small it can be ignored. In this case all signals which would not produce more than 0.1 V change in a load or source terminal voltage have been dropped.

There will be times when two (or more) signals arrive simultaneously at line to line junctions as in point P of Fig. T-19.

$$t = 3\ell_A \delta_A + 2\ell_B \delta_B$$

and there are two paths by which signals can arrive at this point at this time. Consequently, the continuing waves must be arrived at as the sum of two components. Here the wave continuing to the right is the sum of the reflection $(+0.208 \text{ v})(\rho_{BA})$ and the continuing wave $(+0.318 \text{ v})(1+\rho_{BA})$. There are often more than two lines coming together at one point and in that case there may be more than two signals arriving simultaneously at such a point. The waveforms resulting from this example are shown in Fig. T-20.

$R_{0B} = 120 \, \Omega$, $\delta_{1B} = 1.25 \, \text{ns/ft}$, and is 23 feet long. The excitation is a 10 V step applied at $t = 0$ through $100 \, \Omega$ and $R_L = 250 \, \Omega$ as in Fig. T-16. Let us find the input and output waveforms.

In constructing the reflection diagram some care must be exercised so that the information generated is meaningful. First consider the delays in each line. The multiple signals on such a system are of common origin and have a common origin in time. Therefore, their physical relationship in the reflection chart should be consistent with their temporal relationship since time is displayed vertically in the reflection diagram. Consider Fig. T-18 showing a signal originating at the left hand line end

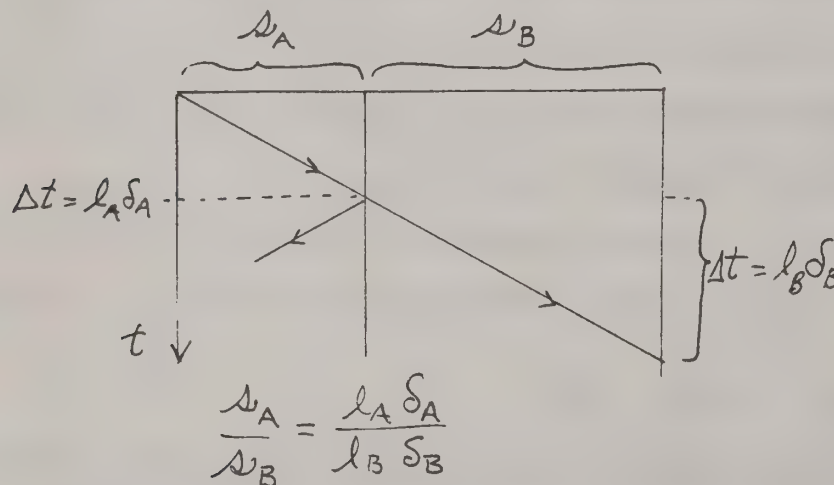


Fig. T-18. Constructing the reflection diagram.

which finally reaches the right hand end of the line. The delays encountered are $t_A = l_A \delta_A$ and $t_B = l_B \delta_B$. If these delays are to be properly displayed along the $-y$ axis, then the distances along the $+x$ axis must be proportioned so that

$$\frac{s_A}{s_B} = \frac{\delta_A l_A}{\delta_B l_B} \quad (\text{T-43})$$

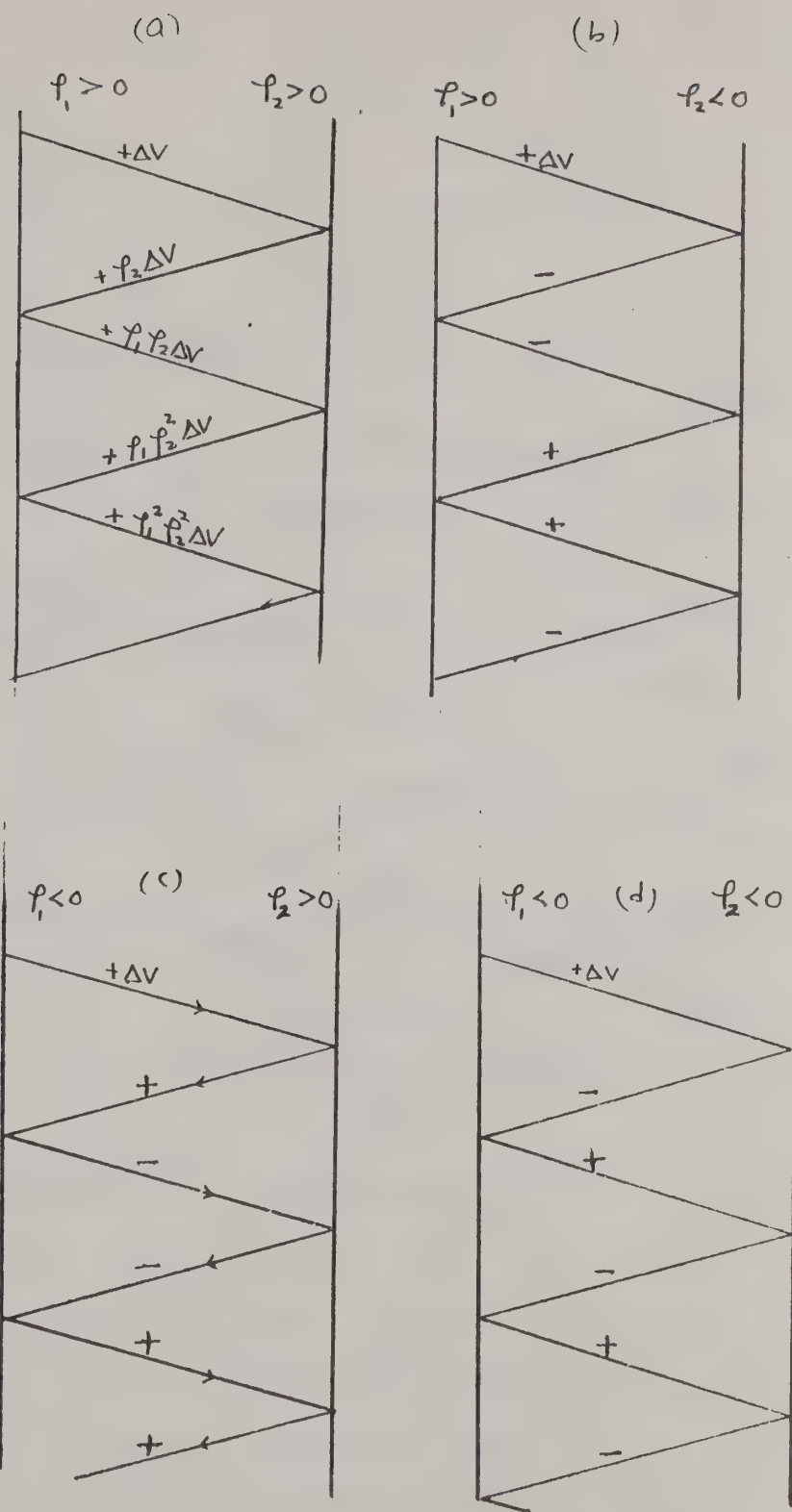


Fig. T-14. Reflection diagrams for mismatched lines

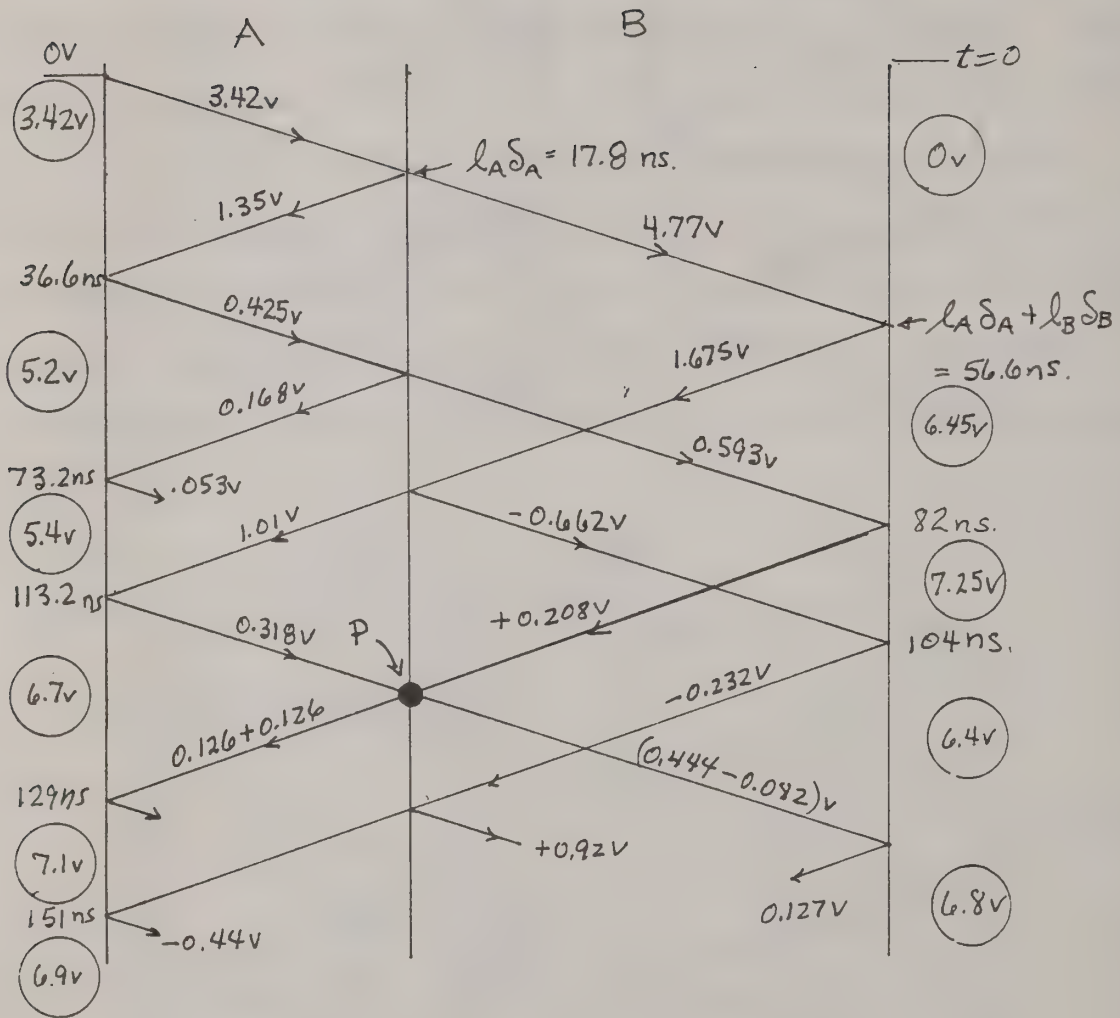


Fig. T-19. Reflection diagram for example T1

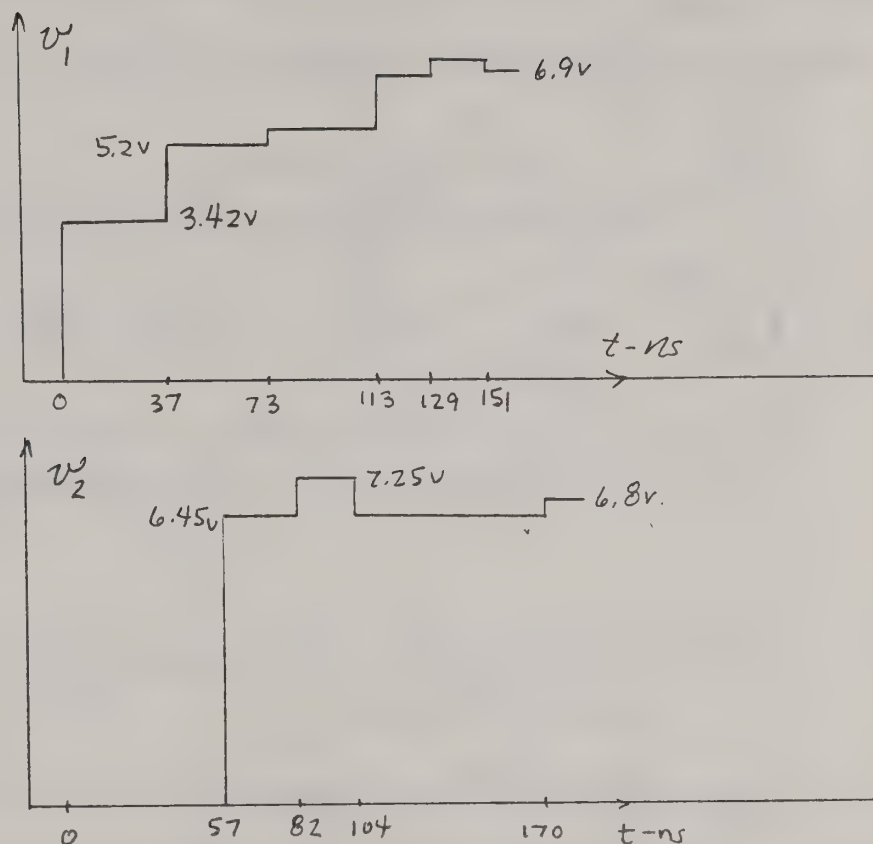


Fig. T-20. Input and output voltages for Example 1.

T1.7. CAPACITIVE AND INDUCTIVE LOADS

Consider the situation in Fig. T-21a where a transmission line has a capacitive load. In practice the capacitor is only part of the load, but by studying this case the response of a partly capacitive load may be understood better.

Initially the source does not "see" the load and the input signal is that obtained if the source faced a resistance R_0 . After a delay of $t = 2\delta$ the step wave signal reaches the load end of the

line. Using the equivalent circuit of the line developed previously, we find a signal twice the amplitude of the incident wave delivered through a resistance R_0 as shown in Fig. T-21. This is a simple transient problem producing a signal v_2 which

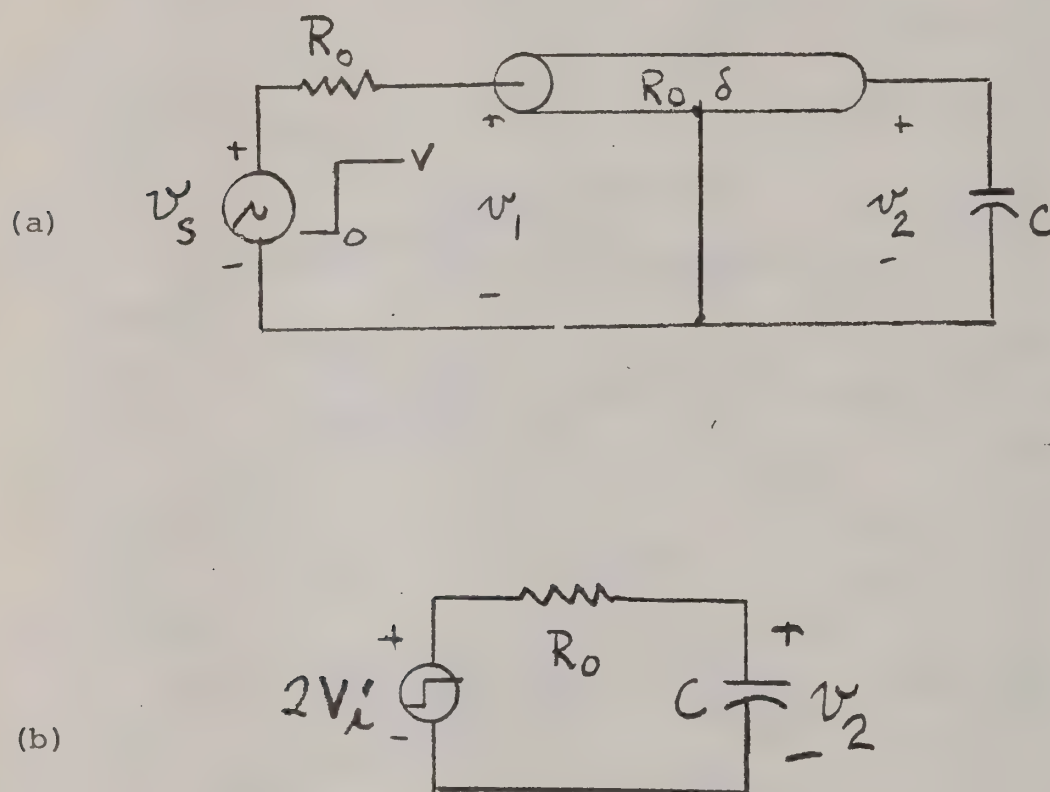


Fig. T-21. a. Transmission line with capacitive load
b. Equivalent circuit at the end of the line
at $t = l\delta$

is an exponential rising from 0 to a final value V with time constant $R_0 C$. If the source resistance were not equal to R_0 there would be other signals due to reflections arriving at the receiving end. In this case, however, the above equivalent circuit determines the output voltage, v_0 . Since a step wave of amplitude V is delivered, the exponential may be written as

$$v_2 = VU(t-l\delta)(1-e^{-(t-l\delta)/\tau}) \quad (T-44)$$

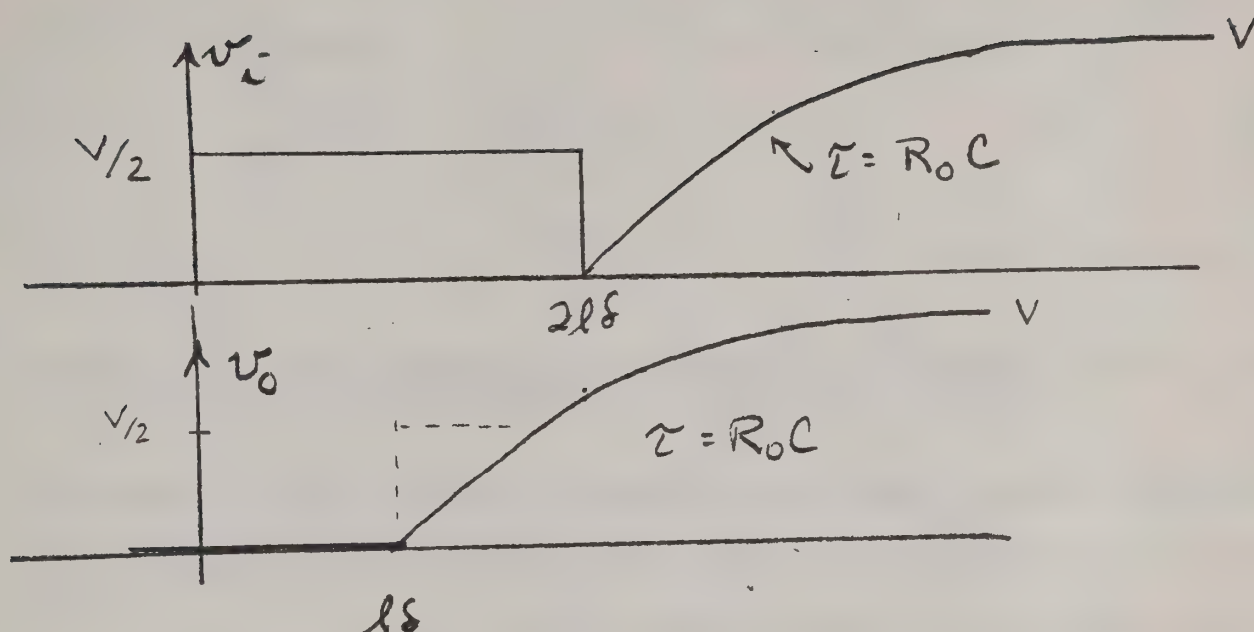


Fig. T-22. Step response of a transmission line with a capacitive load

The difference between the incident signal and the load signal must be the reflected voltage which proceeds back toward the source end. Accordingly, this reflection arrives at the source end of the line at $t = 2\ell\delta$ and since the source end is terminated in its characteristic impedance, no reflections are observed. The left hand end exponential rise is a reproduction of that occurring at the right hand end. Had there been a reflecting termination at the source end we would have a more complicated problem on our hands.

The second example is a transmission line driving an inductor at the load end as indicated in Fig. T-23. Assume as before that the source end is driven from $R_s = R_0$. As before, the source does not "see" the inductor until $t = 2\ell\delta$, and the input signal v , is simply a step wave in response to the step wave of the generator.

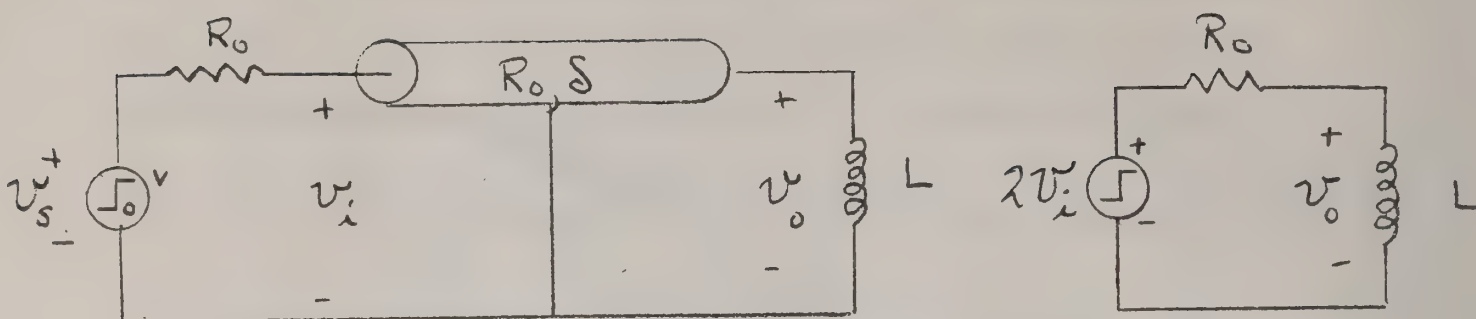


Fig. T-23. Transmission line with an inductive load. (a) circuit diagram (b) equivalent circuit at the load at $t = l\delta$

When the incident step wave arrives at the load end, the line appears to the inductor as a step wave source of $2v_{\text{incident}}$ in series with R_0 as shown in Fig. T-23b.

The response of the inductor to this signal is an exponential

$$v_2 = VU(t-l\delta) t^{-(t-l\delta)/\tau} \quad (\text{T-45})$$

where $\tau = L/R_0$.

The signal seen at the load end in response to the incident square wave of amplitude $V/2$ is shown in Fig. T-24a. The difference between the actual response and the incident signal again must be the reflection. Accordingly, at $t = 2l\delta$ the input voltage jumps by $V/2$ and then exponentially falls in the same manner as v_2 as shown in Fig. T-24b. Since $R_s = R_0$ there are no reflections.

What should have been learned from these two examples? The first point is the essential nature of non-resistive terminations. Capacitors appear as short circuits to fast signals while inductors appear as open circuits. The second is that the presence of such elements on a transmission line can produce large reflections. The nature and distance down the line of this termination or imperfection

can be determined from an examination of the transmitting end waveforms. Third, the transmission line always appears as a resistance, R_0 to any such element.

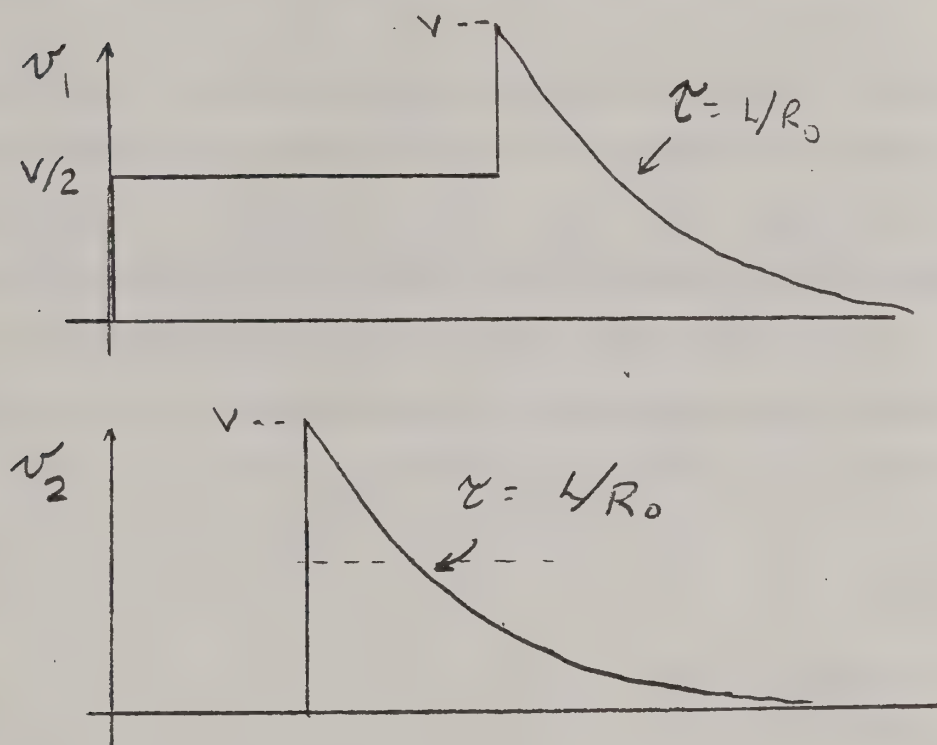


Fig. T-24. Step response of a transmission line with a capacitive load. (a) input voltage (b) load voltage

T1.8. SHORT LINES

If the transmission line is very short, the reflections occur rapidly. For the case where the line is lightly loaded on the load end of the line and lightly loaded on the sending end, a waveform like that of Fig. T-15a (for $R_S > R_0$ and $R_L > R_0$) will occur and the waveform appears to be approximately an exponential. Similarly if $R_S < R_0$ and $R_L < R_0$ and the line is short the

response will be like that of Fig. T-15d and again an exponential recovery is evident. It can be shown that in the case of the lightly loaded or the heavily loaded short line that the line may be approximated by an inductance or a capacitance. In such case the response to the input signal will be exponential in nature.

For example, consider the line to be driven by a generator with a high output resistance and loaded on the other end with a CMOS or MECL gate where $R_L \gg R_0$. The line appears to be predominantly capacitive as shown in Fig. T-25a so that the response is approximately that shown in Fig. T-26a. There appear to be a number of different approximations in the literature for this case. One approximation^{*} is that the time constant is given by

$$\tau = \frac{2\ell\delta}{\ln\rho_1\rho_2} \quad (\text{T-46})$$

Another approximation is that

$$C_x = \frac{\ell\delta}{R_0} \left(1 - \frac{R_0^2}{R_S^2}\right) \quad (\text{T-47})$$

if $R_S = R_L$.

Similarly, the shorted line or the line with $R_S < R_0$ and $R_L < R_0$ gives a response like that of Fig. T-25(b) with the line appearing as an excess inductance L_x , given by^[2]

$$L_x = \ell\delta R_0 \left(1 - \frac{R_S^2}{R_0^2}\right) \quad (\text{T-48})$$

if $R_S = R_L$.

^{*} R. E. Mattick, "Transmission Lines for Digital and Communication Networks", McGraw Hill.

² Millman and Taub, "Pulse Digital, and Switching Waveforms, McGraw Hill, New York 1965.

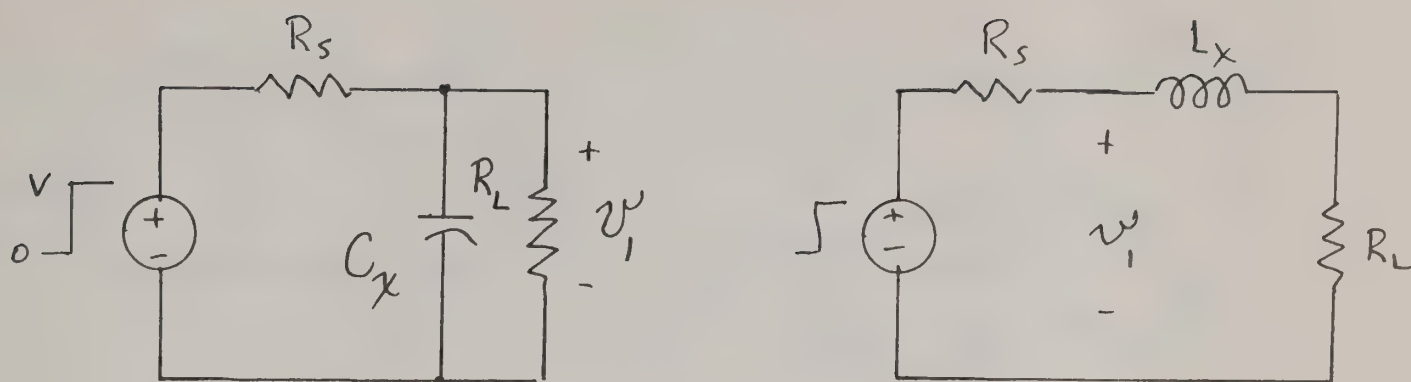


Fig. 25. (a) Equivalent input circuit for a short line with $R_L > R_0$ and $R_S > 0$. (b) Equivalent input circuit for a short line with $R_L < R_0$ and $R_S < R_0$

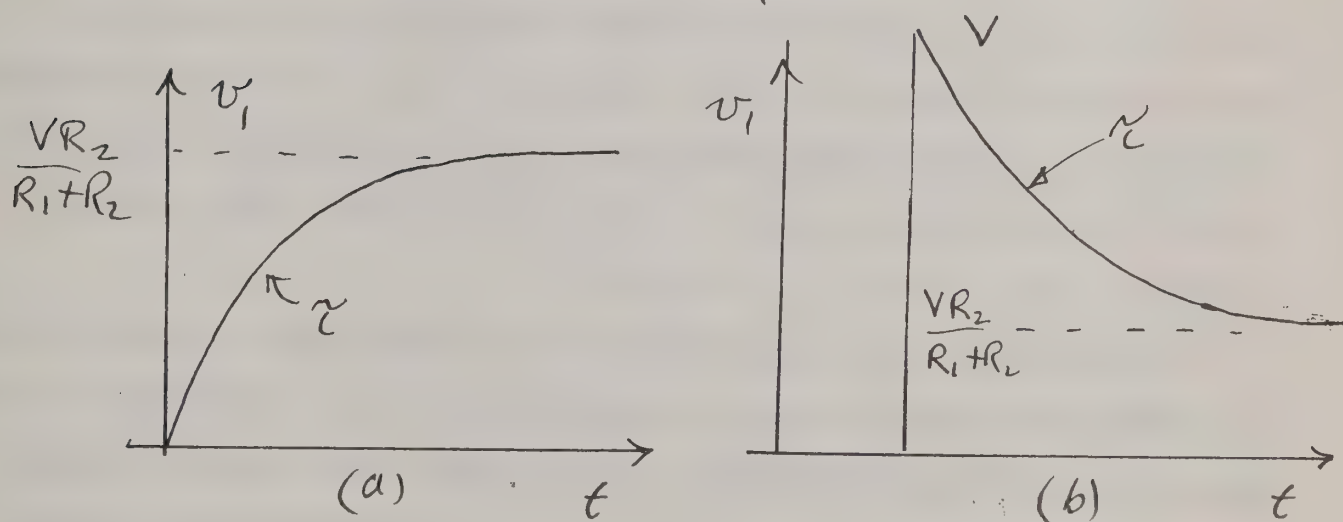


Fig. 26. Input of the short line for the two cases of Fig. T-20.

T1.9. NON-STEP INPUTS

To aid the analysis of transmission line transients involving non-step inputs we may visualize an input signal as being comprised of a number of small steps. For example, consider the case shown in Fig. T-27(a) where a ramp signal is applied to a transmission line. The ramp, can be viewed as a sequence of equal steps. The number of steps can be adjusted to provide the required accuracy.

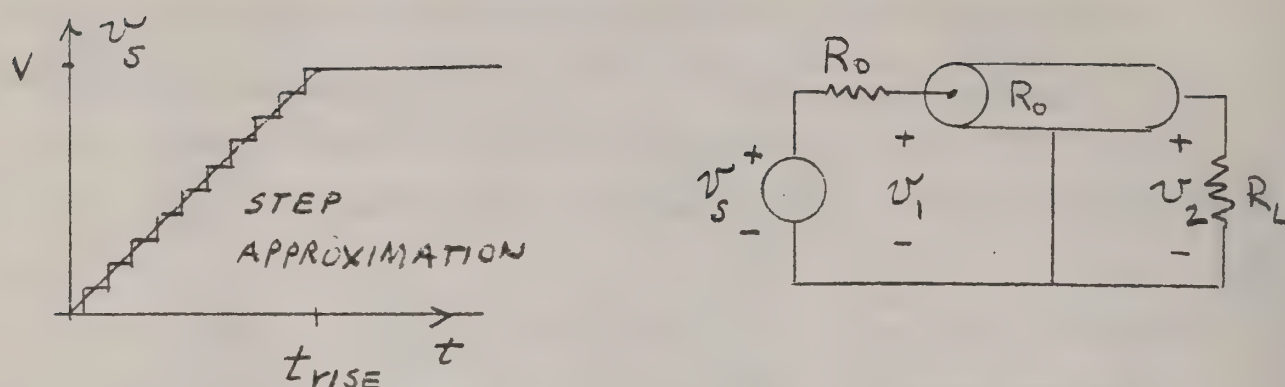


Fig. T-27. Transmission line with ramp input (a) ramp signal
(b) circuit

More complicated waveforms can be thought of as summations of simple waveforms with the response being the summation of the responses to the individual simple waveforms. This will give the correct analysis for linear source and load components. With non-linear source and/or load some results may be obtained by careful application of these ideas.

To continue this example consider the case shown in Fig. T-27(b) where the ramp signal is applied through R_0 to a transmission line terminated in a resistance, R_L . As a first case assume that the line is of such length that $2\ell\delta > t_{\text{rise}}$ and examine the input signal, v for the case where $R_L = \infty$. Actually the approximation of the ramp as a series of steps is unnecessary if we can visualize the delay and reflections of various parts of the signal waveform. The start or termination of the waveform are delayed just as a step occurring at that time would be. The step reflections are a fraction, ρ_2 of the incident step approximations. As a result the entire incident signal waveform is reflected by this factor ρ_2 . Consequently, the step approximations need not be made, they were introduced here as an aid to understanding the phenomenon.

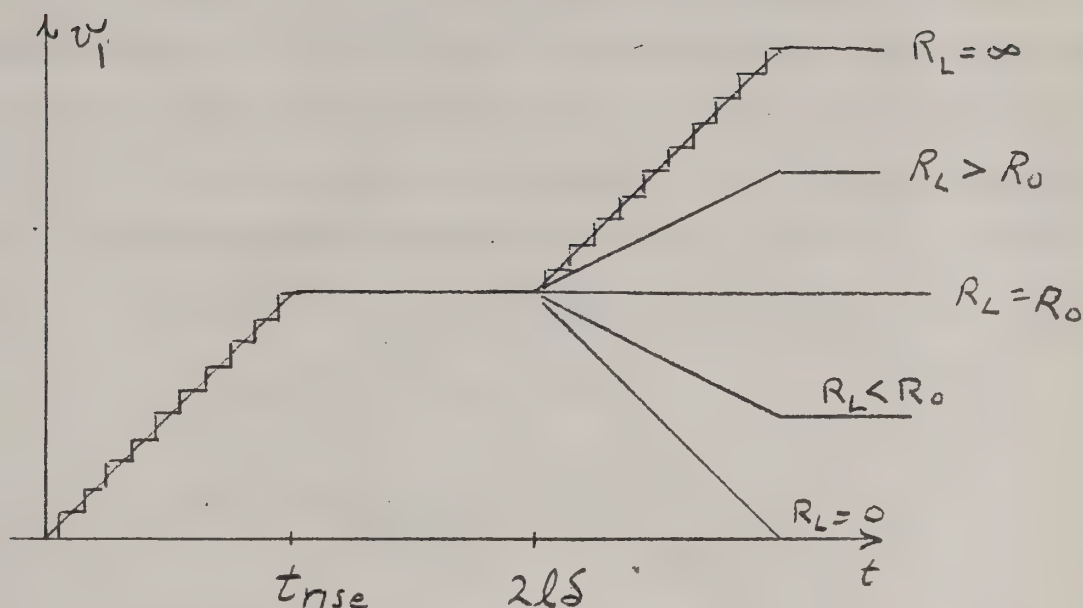


Fig. T-28. Response of a transmission line to a ramp input signal for various terminations. In this case $2l\delta > t_{\text{rise}}$

The input signal v_i is shown for a range of R_L . Notice that for $R_L = R_0$ there are no reflections. The final value of v_1 ranges from 0 for $R_L = 0$ to V for $R_L = \infty$. The waveshape of v_2 is identical to that of the applied signal, v_s , while the waveshape of v_1 is not.

As a second case assume the line to be short so that $2l\delta < t_{\text{rise}}$. Now the reflections return in time to be added to or subtracted from the applied signal. The waveform of v_1 is altered during the rise of the signal. It is interesting to note that again the signal at the load end (v_2) is a replica of the generator signal, while the line input signal (v_1) is not.

For non-linear terminations care must be exercised. For instance, assume that the load, R_L , of the preceding example is

replaced by a silicon diode from line end to ground. Let the input be a 0 to 1 volt ramp. To simplify the example assume the diode clamps the output terminal at 0.6 volts. Now, in this example, the load termination is initially an open circuit during the first part of the input signal rise. However, when the voltage at the R.H. line end reaches 0.6 V the termination changes from an open to a short circuit. This results in a reflection coefficient

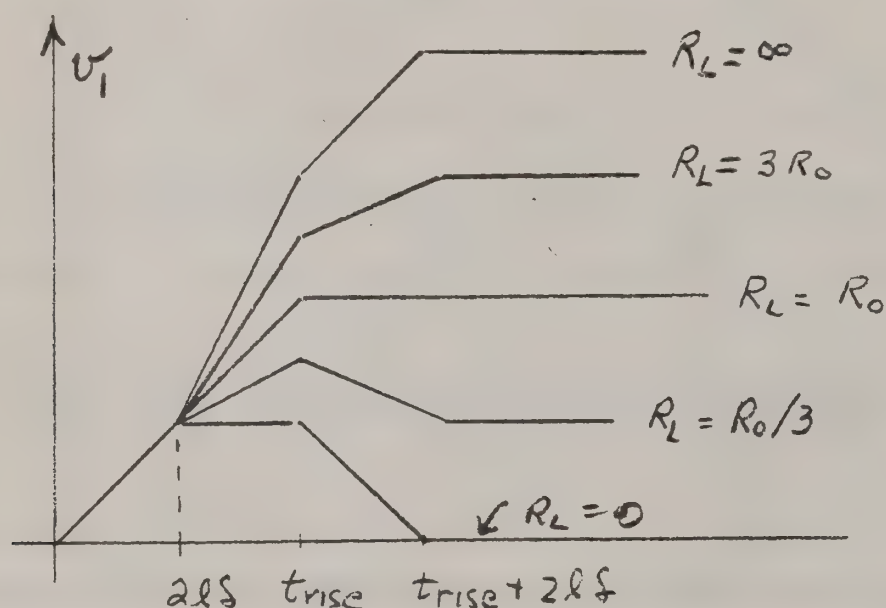


Fig. T-29. Response of a transmission line to a ramp input signal for the case where $2\delta < t_{rise}$

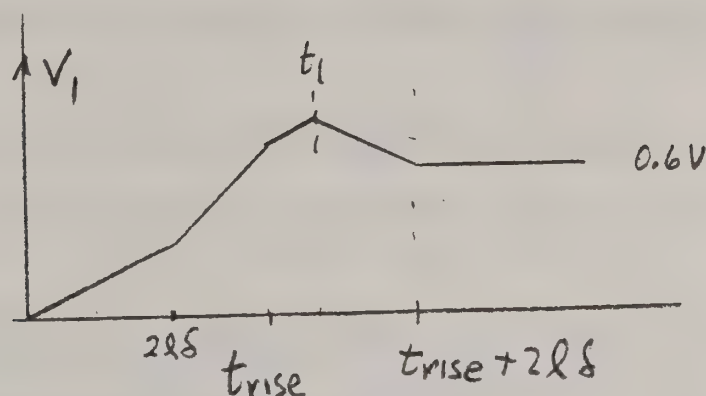
$\rho_2 = -1$ causing any signals arriving after v_2 reaches 0.6 V to be inverted and sent back undiminished in amplitude to the source. The resulting input and output waveforms are shown in Fig. T-30.

If the reflection returns during the rise of the input signal, the load is effectively "seen" by the generator. A line is said to be of critical length if one round trip (2δ) is exactly equal to the rise time of the incident signal.

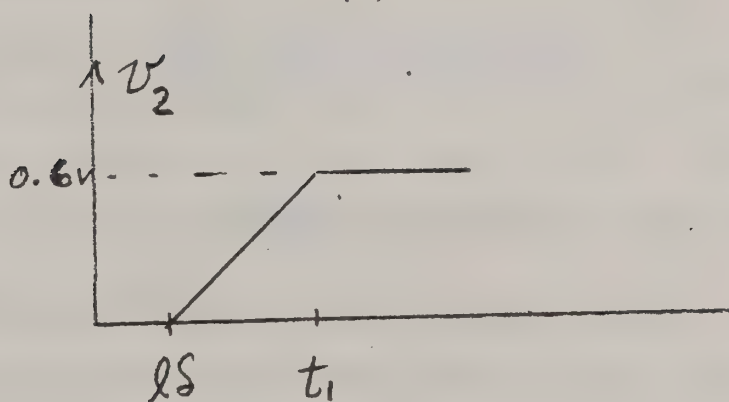
$$\text{Critical length} = \ell_c = t_{rise}/2\delta$$

(T-49)

This defined length is used as a benchmark to separate long lines and short lines. If $l \gg l_c$ obviously the line is long and vice versa.



(a)



(b)

Fig. T-30. Waveforms for a diode termination example

T2.0. GRAPHICAL ANALYSIS OF TRANSMISSION LINE TRANSIENTS

The graphical solution of transmission line transients is based on the graphical solution of simultaneous equations. The voltage-current characteristics of source, line and load all must

be known - either in analytic or graphical form. The solution is first obtained in the I-V plane and from that solution the time domain viewpoint is constructed. Although the characteristics of load and source need not be linear, the first example will have resistive terminations in order to simplify the graphical construction.

Figure T-31 shows the circuit diagram of a transmission line excited by a long pulse. From earlier discussions it is known that

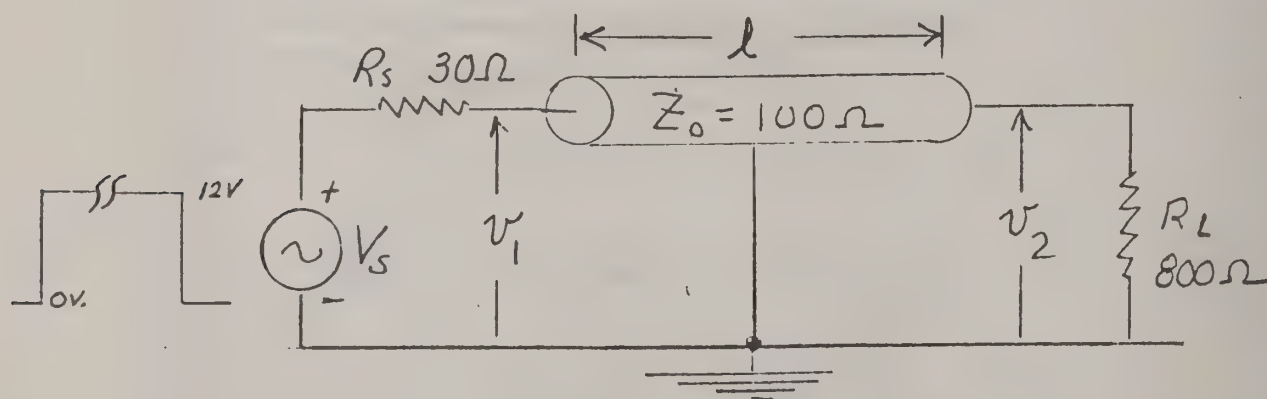


Fig. T-31. Example for graphical solution

the equivalent circuit looking into the line during the initial transient is that of Fig. T-6a where the line is represented by a resistance R_0 in series with a voltage source $V_{L0} - I_{L0}R_0$. In this example the initial conditions are such that $V_L, I_L = 0$ as indicated in Fig. T-32. Graphical solution of this circuit can be accomplished by plotting on the same I-V plane the curve

$$v_1 = i_1 R_0, \quad (T-50)$$

representing the transmission line, and

$$v_1 = 12 - i_1 R_s \quad (T-51)$$

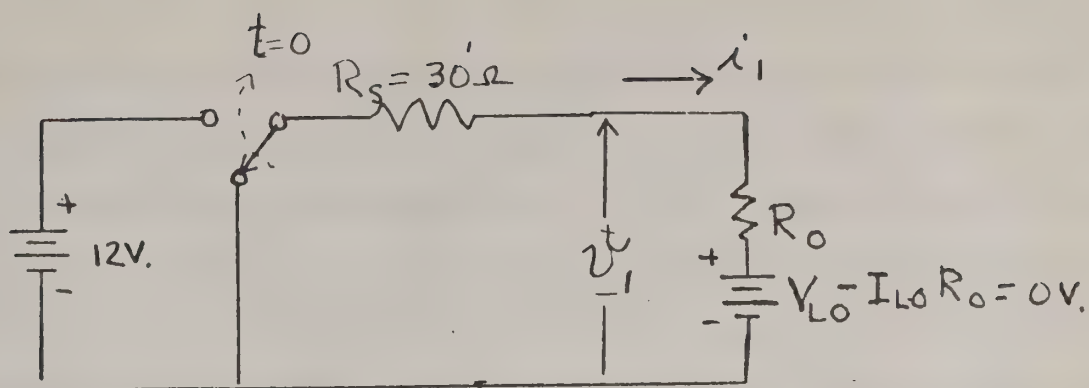
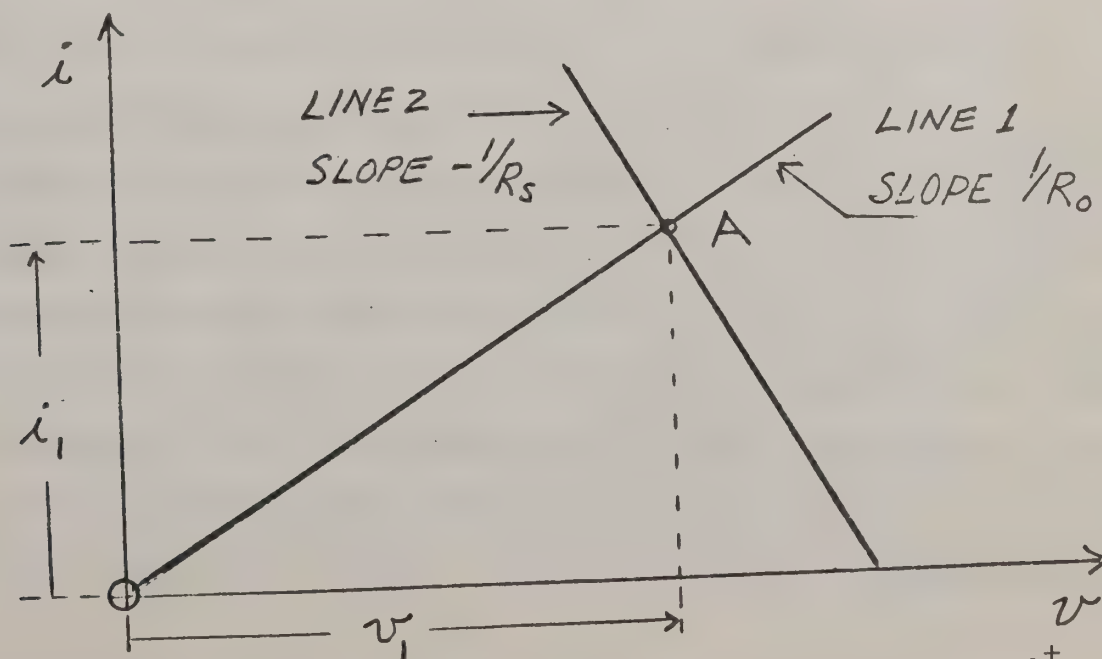


Fig. T-32. Initial input transient

representing the source. Where these two lines intersect, both equations are satisfied and the new value of v_1 and i_1 are obtained. This graphical solution is shown in Fig. T-28. The source is represented by line 2 while the transmission line is represented by line 1. Their intersection at point A provides the current and voltage into the line at the instant that the switch occurs. Since initially $v_1 = 0$ and $i_1 = 0$, the values of v_1 and i_1 obtained here describe the incident voltage and current waves launched on the line.

Fig. T-33. Initial transient voltage at $t = 0^+$

This wave travels down the line and eventually reaches the load. Recalling earlier discussions, the line at the instant that the wave reaches the load can be represented by an equivalent circuit having a voltage source $2v_{inc.}$ in series with a resistance R_0 as shown in Fig. T-34. This equivalent circuit is valid for the instant $t = \ell\delta$. A graphical solution for the line output voltage is obtained from the circuit of Fig. T-34. The solution is obtained

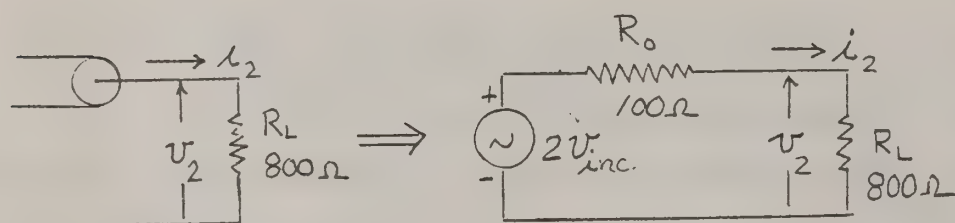


Fig. T-34. Equivalent circuit at load at $t = \ell\delta$

by plotting

$$v_2 = i_2 R_L \quad (T-52)$$

and

$$v_2 = 2v_{inc} - i_2 R_0 \quad (T-53)$$

as shown in Fig. T-35. Line 4 of Fig. T-35 represents the load while line 3 represents the transmission line. Note that the line 3 intercepts are $2v_{inc}$ and $2i_{inc}$ and that line 3 has a slope $-1/R_0$. Note also that line 3 passes through point A. The simultaneous solution provided by lines 3 and 4 occurs at point B which is the receiving end voltage and current at $t = \ell\delta$. The output voltage v_2 is the line voltage and is the sum of incident and reflected signals.

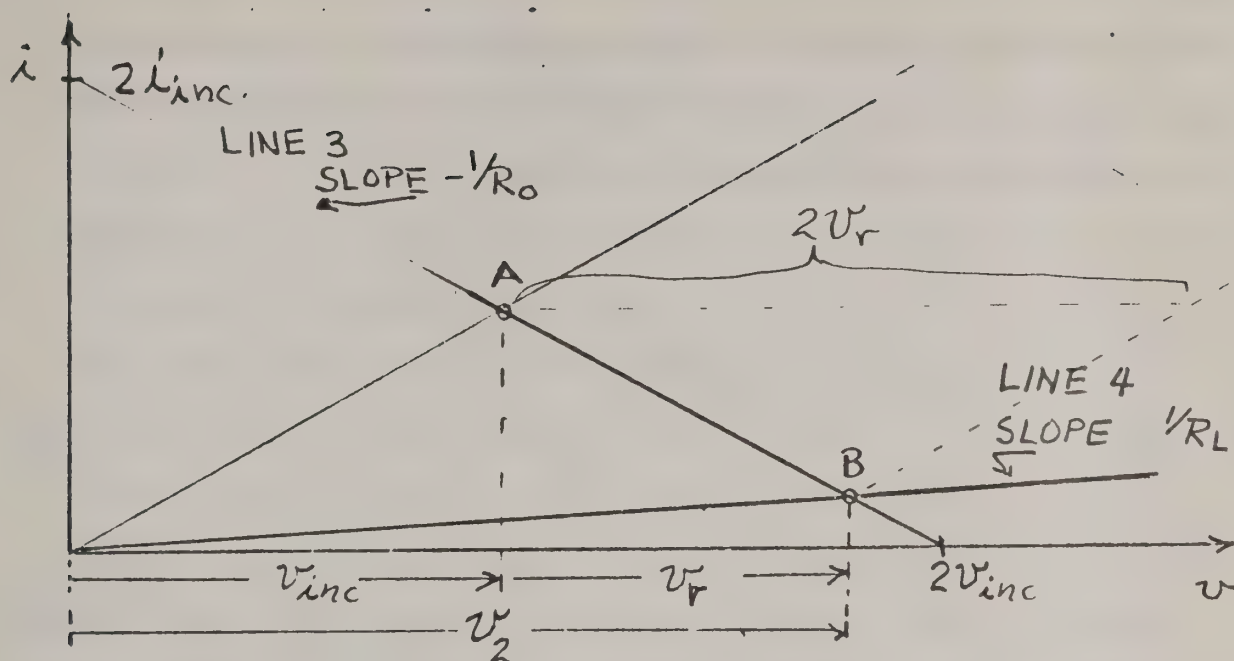


Fig. T-35. Solution at receiving end at $t = L\delta$

The signals at this point have been fairly easy to follow. However, the voltage at point B is the sum of two components and the next point to be determined will have three components, etc. The voltage v_r becomes the signal incident at the left hand end at $t = 2L\delta$. The same construction technique may be used, but with the added complication that the voltage at each end is the sum of all previous waves arriving at or produced at that point. The discussion of the solution becomes more complicated. However, since the rationale has been established, let us detail the mechanism for generating additional signals without the elaborate justification of each step. The construction procedure is carried out in Fig. T-36 and is outlined as follows:

1) Start at the v - i point representing the initial line conditions.

Draw a straight line of slope $1/R_0$ through this point (Line 1).

- 2) As the input changes a new v-i line representing the input should be drawn (Line 2). The intersection of this curve and the one drawn in (a) is point A and gives the initial voltage and current into the transmission line.

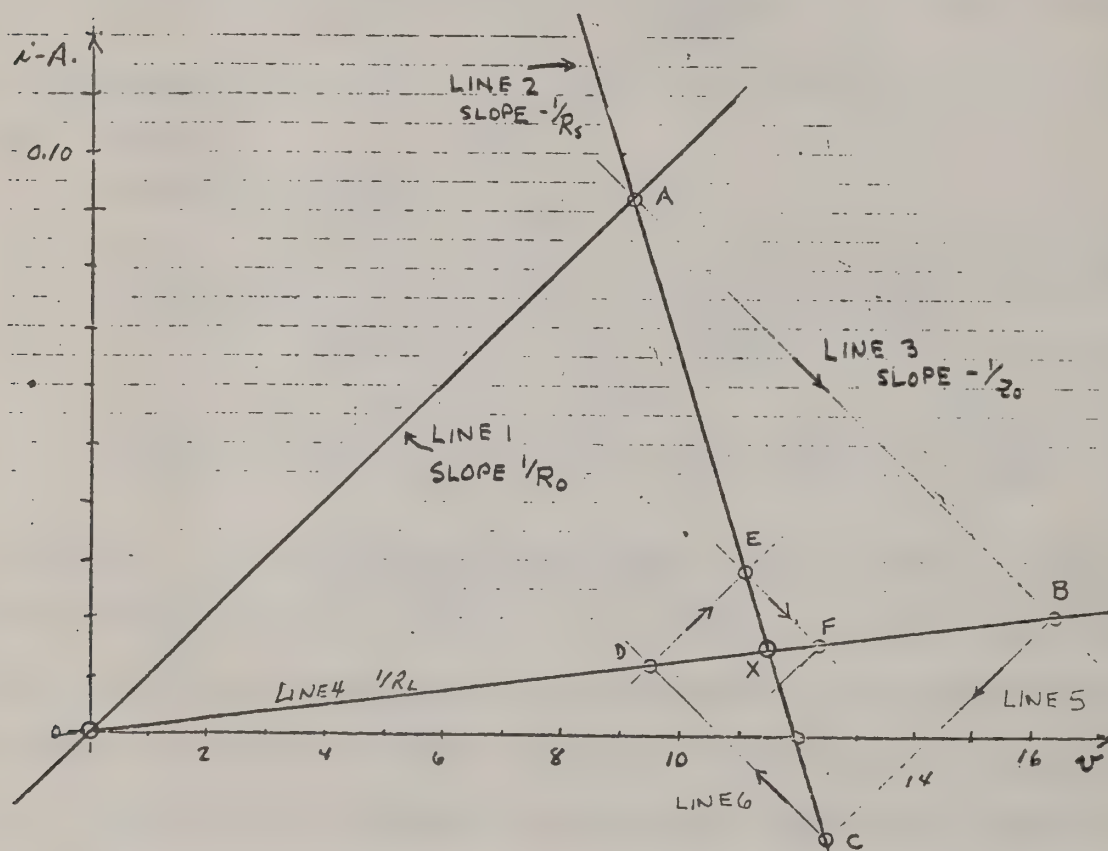


Fig. T-36. Example graphical solution

- 3) Draw a straight line through point A at slope $-1/R_0$ (Line 3).
- 4) The intersection of the straight line just drawn and the curve representing the load (line 4) provides point B and the transmission line output voltage and current.
- 5) Through B draw a straight line (line 5) of slope $1/R_0$. The intersection of straight line 5 and the curve representing the source (line 1) provides a point C at the new value of source voltage and source current.

- 6) Through C draw a straight line (line 6) of slope $-1/R_0$. The intersection of straight line 6 and the line representing the load gives point D at the new value of load voltage and load current.
- 7) Repeat steps 5 and 6 through successive points until a steady solution is obtained (point X).

Note that points A, C, E, G, etc., give the transmission line input voltages at times $t = 0, 2\delta, 4\delta, \text{etc.}$ Similarly points B, D, F, H, etc., represent load voltages at times $T = \delta, 3\delta, 5\delta, \text{etc.}$ The plots of $v_1(t)$ and $v_2(t)$ are obtained from these values. Figure T-37 shows transmission line input and output voltages for this example. The ultimate convergence of the spiral of Fig. T-36 is at the intersection of source and load I-V curves. When all transients have died out (at $t = \infty$) the line, being lossless, should be a short between load and source.

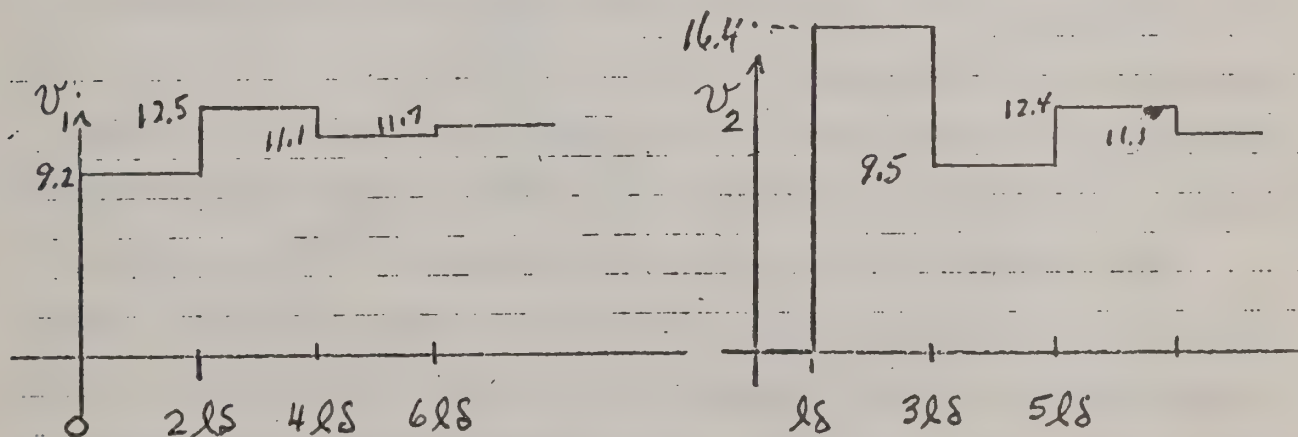


Fig. T-37. Input and output voltages

Next, let V_s change from the steady state value just obtained back to zero. The initial state of the transmission line is described by the constant voltage and current values V_{L0} and I_{L0} . These

are the line voltage and current at point X. The line appears as if it were a voltage source of value $V_{L0} - I_{L0}R_0$ in series with a resistance R_0 . Graphically this is expressed by a straight line of slope $1/R_0$ through point X. This is shown as line R1 in Fig. T-38. The intersection of the curve representing the source and line R1 which represents the transmission line is point M and provides the signal voltage and current into the line the moment after the source change takes place.

As before, draw a straight line of slope $-1/R_0$ through point M. This straight line intersects the load line at point N and gives the voltage and current at the receiving end of the transmission line at time 2δ after point M conditions are established. As before continue to draw straight lines of slope $1/R_0$, $-1/R_0$ intersecting (alternately) load and source curves and points P, Q, R, S, T, etc. will result. The final values of voltage and current will be at point Y where load and source characteristics intersect (ultimately the transmission line is a very low d.c. resistance connecting load and source.) As before, the voltage (or current) waveform can be established from these values found graphically.

This example provides the basic mechanism for the graphical analysis of transmission line transients. Some additional complications may arise. For example, the initial conditions may be such that $V_{L0} \neq 0$ and $I_{L0} \neq 0$. The initial graphical representation of the transmission line will be by a line of slope $1/R_0$ through the point V_{L0} , I_{L0} just as in Fig. T.38. A very careful initial definition of voltages and currents and their polarities will be helpful in obtaining correct solutions.

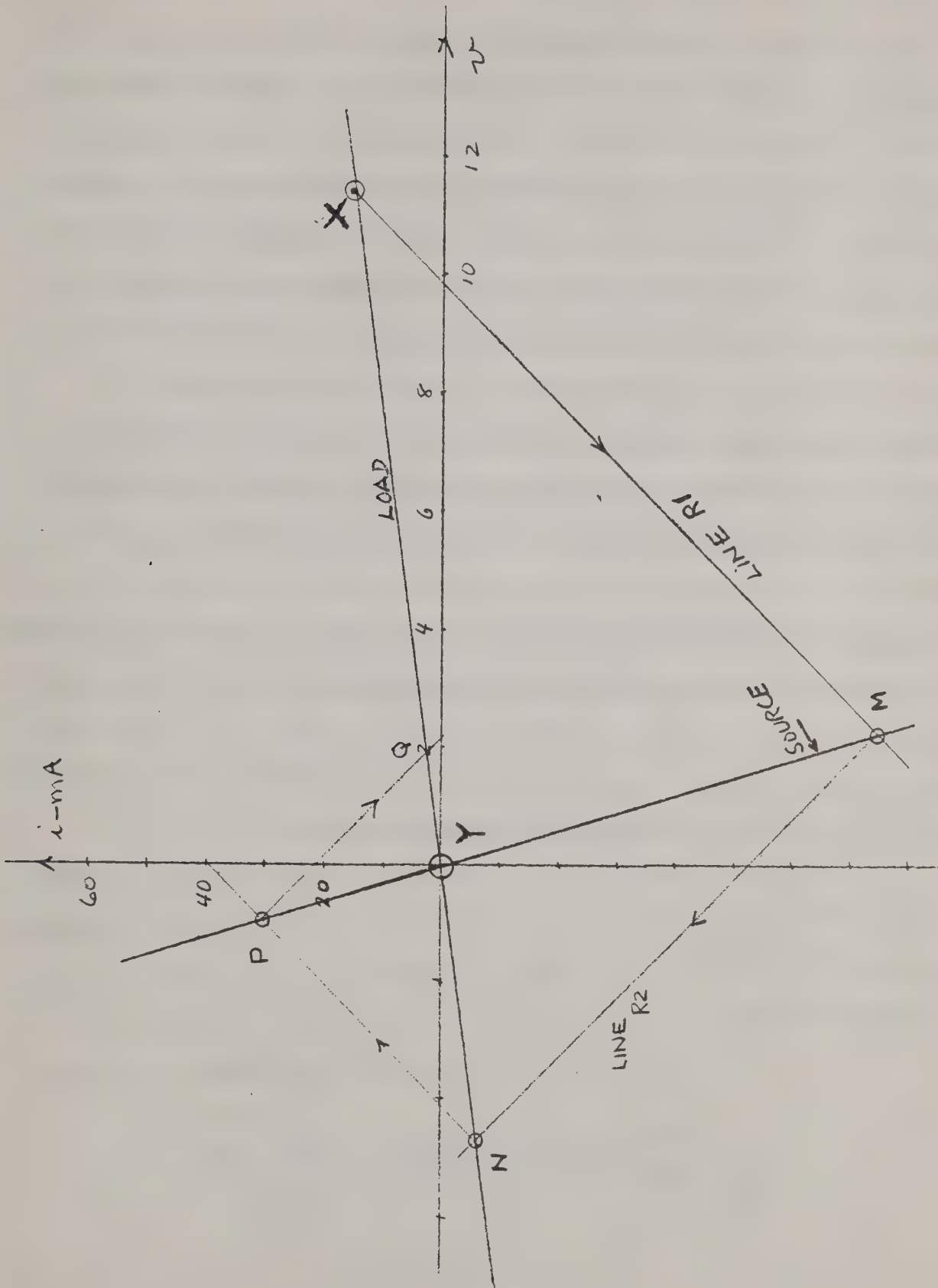


Fig. 38. Graphical solution of return transient

T2.1. NON-LINEAR SOURCE AND LOAD

Another complication frequently present is that neither load nor source are linear. This does not alter the graphical analysis procedure. As noted initially, a plot of the I-V characteristics of source and load are needed. These characteristics may be linear or non-linear. These characteristics can be obtained from whatever analytic expression, data points, etc. are available. However, it is wise to very carefully define voltages and currents and their polarities in order to avoid errors. The circuit in Fig. T-39 provides a non-linear example illustrating the points mentioned. In this example a transistor turns on and off driving a load at the end of a $100\ \Omega$ transmission line. Both load and source are non-linear.

First, a definition of voltage and current is made as indicated in Fig. T-39. This convention decides the positioning of the load and source characteristics on the I-V plane. Next, find the characteristics of the load. One can write an equation or reason out the characteristics. For example, the equations are:

$$i_2 = i_R + i_D \quad (T-54)$$

$$i_R = -(5 - v_2) / 1k\Omega \quad (T-55)$$

$$i_D = -f(-v_2) \quad (T-56)$$

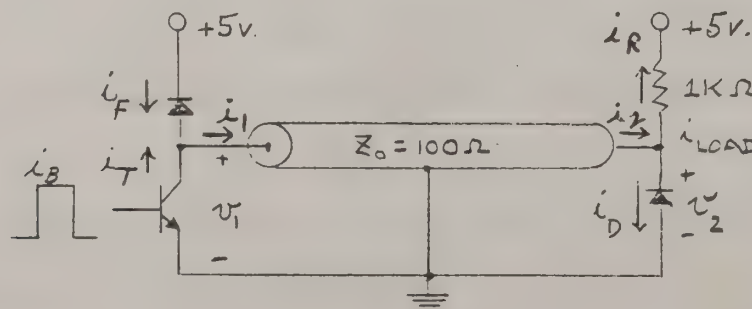


Fig. T-39. Non-linear source and load

The equation for i_R is related to that of the familiar load line. The sign reflects the chosen polarity for positive i and v . The diode current i_D is specified to be the diode function of the voltage $-v_2$. In the steady state $i_2 = i_1$ and $v_2 = v_1$. The load current i_2 is described by the sum of the two currents i_R and i_D at any value of voltage. This is shown in the I-V plane in Fig. T-40. Similarly the source characteristic is the sum of a diode current plus a transistor current,

$$i_1 = i_F + i_T \quad (T-56)$$

However, the transistor current, i_T , is the negative of the collector current of a CE amplifier. Notice that since the transistor has two states (on and off), there are two different source characteristics. The "off" transistor characteristic is virtually coincident with the axis until the voltage v rises above 5V. The current, i_F , is the negative of the diode forward current

$$i_F = -i_{\text{DIODE}} = -f(v_1 - 5)$$

Once the load and source characteristics are available, the steady state values of i and v are found from points X and Y. The graphical solution for the transient occurring as the transistor turns on is indicated in Fig. T-40. Source voltages are found from the sequence of points X, A, C, E, G, etc. while load voltages are found from points B, D, F, H, etc.

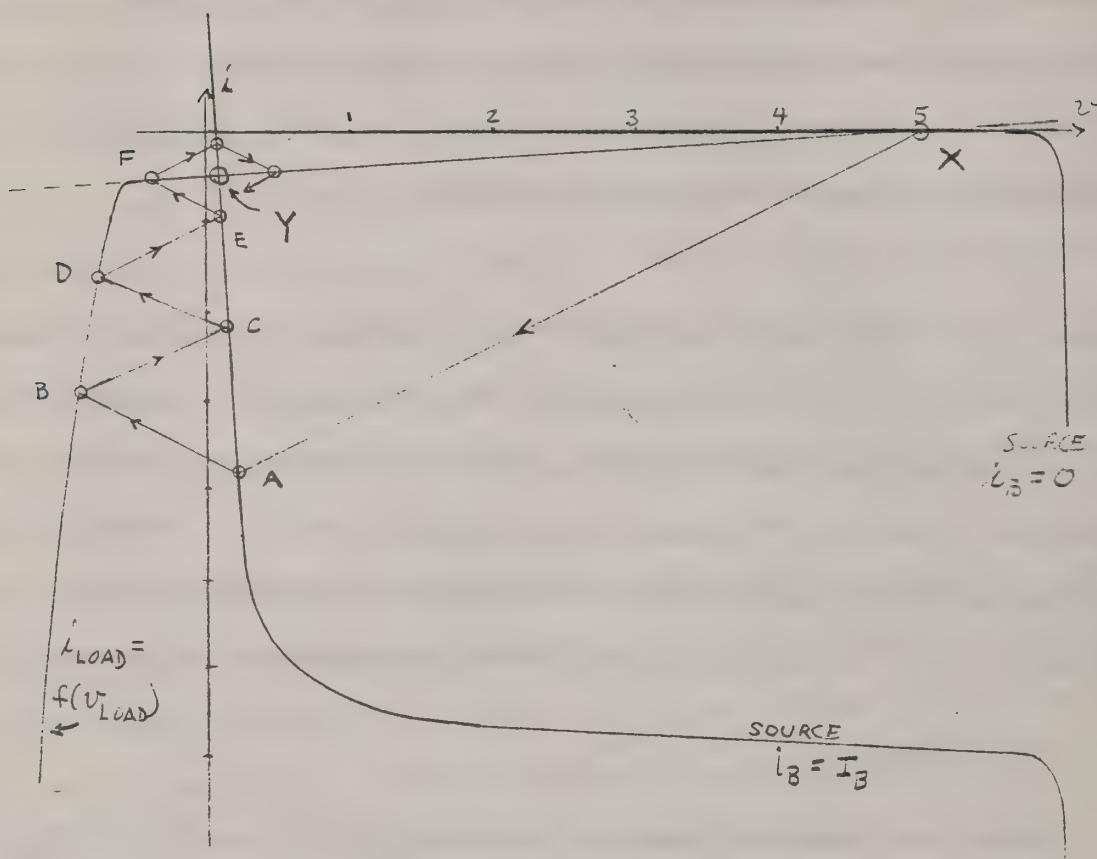


Fig. T-40. Graphical solution of transient

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APPENDIX T-7

Transmission Lines Characteristics

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$R = \Omega/\text{unit length series resistance}$

$G = \Omega/\text{unit length parallel conductance}$

Special case $R \ll \omega L$ $G \ll \omega C$ (or $R/L = G/C$)

$$Z_0 = R_0 = \sqrt{\frac{L}{C}} \text{ the distortionless line}$$

$\beta = \text{phase constant in radians/unit length}$

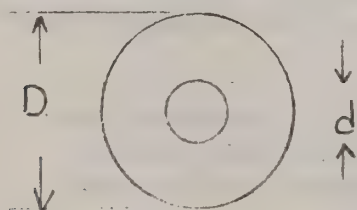
$$\delta = \frac{\beta}{\omega} = \text{time delay/unit length} = \sqrt{LC} = Z_0 C$$

Free space propagation delay = 1.2ns/ft.

Characteristics of common lines:

(A) Coaxial

$$Z_0 \approx \frac{60}{\sqrt{\epsilon}} \ln \frac{D}{d}$$

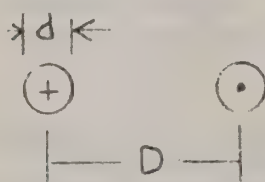


Typical: RG58/U

$$Z_0 = 53.5\Omega \quad C = 28.5\text{pF/ft}$$

$$\delta = Z_0 C = 1.52\text{ns/ft.}$$

(B) Two wire line in air, not near ground:

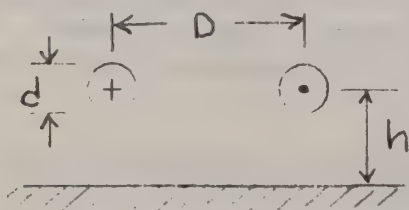


$$Z_0 \approx 120 \ln \frac{2D}{d}$$

Typical: #24 AWG, 1" apart $Z_0 = 552\Omega$

#24 AWG, 1/4" apart $Z_0 = 385\Omega$

(C) Two wire balanced line above ground:

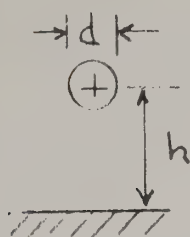


$$Z_0 = \frac{120}{\sqrt{\epsilon}} \ln \left| \frac{2D}{d} \left(\frac{1}{\sqrt{1 + (D/2h)^2}} \right) \right|$$

for $d \ll D, h$

Example: #24 AWG, 1/4" apart, 1/4" above ground. $Z_0 = 289\Omega$

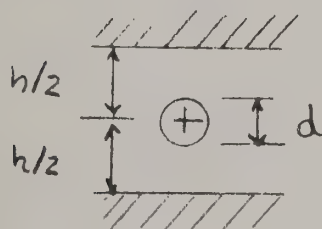
Wire with ground return:



$$Z_0 = \frac{60}{\sqrt{\epsilon}} \ln \frac{4h}{d} \quad \text{for } d \ll h$$

Example: 24 AWG, 1/4" above ground: $Z_0 = 234\Omega$

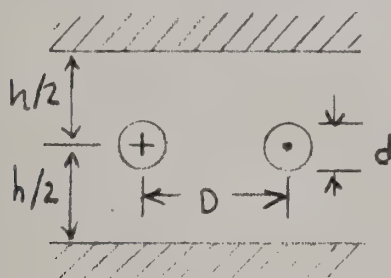
Wire between ground planes:



$$Z_0 = \frac{60}{\sqrt{\epsilon}} \ln \frac{4h}{\pi d} \quad \text{for } \frac{d}{h} < 0.75$$

Example: #24 AWG, 1/4" from each ground plane $Z_0 = 165\Omega$

Balanced line between ground planes:

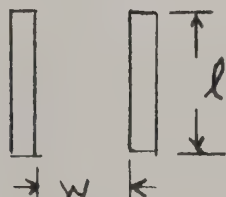


$$Z_0 = \frac{120}{\sqrt{\epsilon}} \ln \left(\frac{4h \tanh \frac{\pi D}{2h}}{\pi d} \right)$$

Example: #24 AWG, 1/4" = D, 1/4" = h/2
 $Z_0 = 280\Omega$

Parallel strip line. For $w/l < 0.1$

$$Z_0 \approx \frac{377w}{l}$$



Example: $l = 3/4"$, $w = 1/16"$

$$Z_0 = 31\Omega$$

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B-1100

THE EXPONENTIAL TRANSMISSION LINE

BY

CHAS. R. BURROWS
Bell Telephone Laboratories

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THE EXPONENTIAL TRANSMISSION LINE
AS AN IMPEDANCE TRANSFORMER.
THEORY AND EXPERIMENT

Presented at
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The Exponential Transmission Line *

By CHAS. R. BURROWS

Bell Telephone Laboratories

The theory of the exponential transmission line is developed. It is found to be a high pass, impedance transforming filter. The cutoff frequency depends upon the rate of taper.

The deviation of the exponential line from an ideal impedance transformer may be decreased by an order of magnitude by shunting the low impedance end with an inductance and inserting a capacitance in series with the high impedance end. The magnitudes of these reactances are equal to the impedance level at their respective ends of the line at the cutoff frequency.

For a two-to-one impedance transformer the line is 0.0551 wavelengths long at the cutoff frequency. For a four-to-one impedance transformer the line is 0.1102 wavelengths long at the cutoff frequency, etc.

The results have been verified experimentally. Practical lines 50 meters and 15 meters long have been constructed which transform from 600 to 300 ohms over the frequency range from 4 to 30 mc. with deviations from the ideal that are small compared with the deviations from the ideal of commercial transmission lines, either two-wire or concentric.

When an exponential line is used as a dissipative load of known impedance instead of a uniform line it is possible to approach more nearly the ideal of constant heat dissipation per unit length. This makes it possible to use a shorter line.

THE exponential line may be defined as an ordinary transmission line in which the spacing between the conductors (or conductor size) is not constant but varies in such a way that the distributed inductance and capacitance vary exponentially with the distance along the line. That is, the impedance ratio for two points a fixed distance apart is independent of the position of these two points along the line. A disturbance is propagated down an exponential transmission line in the same manner as it would be down a uniform line with the additional effect that the voltage is increased by the square root of the change in impedance level and the current is decreased by the reciprocal of this quantity.

The exponential line has the properties of a high pass impedance transforming filter. The cutoff frequency depends upon the rate of

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taper. As the frequency is increased the transfer constant * approaches the propagation constant of the equivalent uniform line. At sufficiently low frequencies the only effect of the line is to connect the input to the load.

Above cutoff the magnitudes of the characteristic impedances at any point are approximately equal to the nominal characteristic impedance * at that point but their phase angles (in radians) differ by an amount which at the higher frequencies is equal to the cutoff frequency divided by the frequency in question. The ratio of input impedance to the input impedance level * of an exponential line terminated in a resistance equal to the impedance level at the output always remains within the range from $1 - f_1/f$ to $1/(1 - f_1/f)$ for frequencies, f , greater than the cutoff frequency, f_1 . For a 2 : 1 transformation this means that the input impedance remains within ± 6 per cent of the desired value for all frequencies above that for which the line is a wave-length long. For a 4 : 1 transformation under the same conditions the irregularities are twice as great.

A transforming network having deviations from the ideal of the order of $\pm (f_1/f)^2$ may be made by connecting an inductance in parallel with the low impedance terminal and a capacitance in series with the high impedance terminal. The magnitudes of these reactances are such that their impedances are equal to the impedance levels of the line at their respective ends at the cutoff frequency. Or expressed in another way the capacitance is equal to $2/(k - 1)$ times the electrostatic capacitance of the line and the inductance is the same factor times the total loop inductance of the line where k is the impedance transformation ratio of the line.

Figure 1 shows the theoretical input impedance-frequency characteristics for 2 to 1 step-up and step-down exponential lines. Curve 1 is for the line with a resistance termination. At low frequencies the input impedance is equal to the load impedance while at high frequencies the line approaches an ideal transformer. Curve 2 is the input impedance of the line terminated with the appropriate resistance-reactance combination. The improvement in the input impedance characteristic for frequencies above the cutoff frequency is evident. At the lower frequencies the input impedance does not approach the terminal reactance but approaches the reactance of the capacitance of the line in parallel with the series terminal capacitance for the step-up line and the reactance of the inductance of the line in series with the shunt terminal inductance for the step-down line. The improvement is not as great as apparent from the figures because the phase angle is

* See appendix for definition of terms.

not improved proportionally. This is easily remedied by completing the impedance transforming network with the appropriate reactance at the input. The resulting input impedance is shown in curve 3. In the "pass" frequency range the maximum reactive component is of

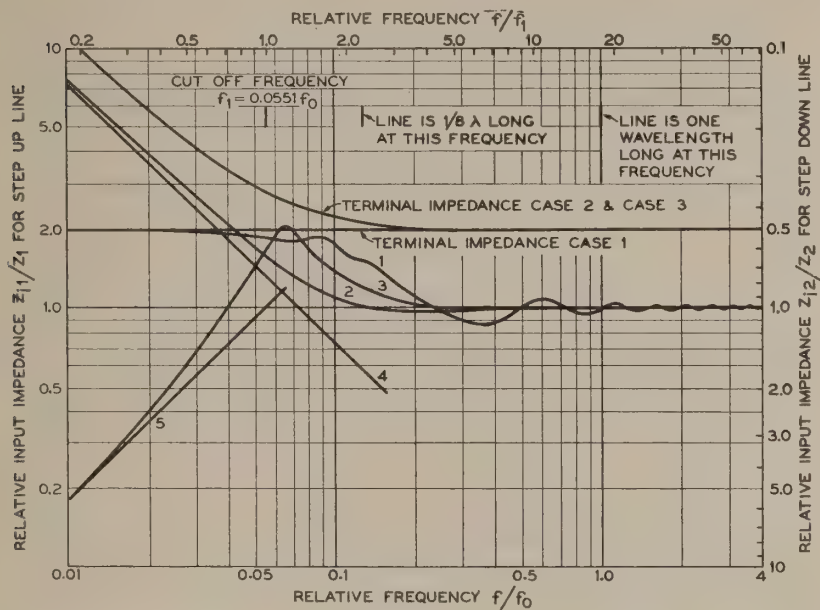


Fig. 1—Input impedance characteristics of 1 : 2 exponential lines. Left ordinate scale refers to step-up line. Right ordinate scale refers to step-down line.

Curve 1—Resistance termination.

Curve 2—With capacity equal to twice the electrostatic capacity of the line in series with the same resistance, $Z_2 = Z_2(1 - jf_1/f)$, for step-up line, or with an inductance equal to twice the total inductance of the line in shunt with the same resistance, $Z_1 = Z_1/(1 - jf_1/f)$, for step-down line.

Curve 3—Termination as for curve 2 with inductance equal to twice the total inductance of the line in parallel with input to the line, $Z_{i1} = Z_1/(1 - jf_1/f)$, for step-up line, or termination as for curve 2 with capacity equal to twice the static capacity of line in series with input to the line, $Z_{i2} = Z_2(1 - jf_1/f)$.

Curve 4—Asymptotic value of impedance of capacity of line in parallel with termination in series with termination for case 2 for step-up line, or asymptotic value of impedance of inductance in series with termination for case 2 for step-down line.

Curve 5—Impedance of shunt inductance added at input for case 3 for step-up line, or impedance of capacity added in series at input for case 3 for step-down line.

the same order of magnitude as the deviation of the impedance from the ideal.

Besides its application as an impedance transforming network, the exponential line may be used as a "resistance" load of constant known impedance that has a high capability for dissipating power. As such it is capable of dissipating more power in the same length of line than

the uniform line. If x is the maximum attenuation in nepers that can be obtained with a uniform line without overheating, the same length of exponential line will have an attenuation of $(e^{2x} - 1)/2$ nepers.

Exponential lines of the proper length have properties similar to half-wave and quarter-wave uniform lines. The input impedance of an exponential line an even number of quarter wave-lengths long is equal to the load impedance times the impedance transformation ratio of the line. When the length of the line differs from an odd multiple of a quarter wave-length by an amount that depends upon the frequency and load impedance, the input impedance is equal to the product of the terminal impedance levels divided by the load impedance.

MATHEMATICAL FORMULATION

The telegraph equations for the exponential line may be solved by the methods employed in the problem of a uniform line. The resulting equations for the voltage and current at any point along the line are

$$v_x = Ae^{-(\Gamma - \frac{\delta}{2})x} + Be^{+(\Gamma + \frac{\delta}{2})x} = Ae^{-(\Gamma - \frac{\delta}{2})x} \left[1 + \frac{B}{A} e^{2\Gamma x} \right] \quad (1)$$

and

$$i_x = \frac{A}{Z_0} \frac{\Gamma - \frac{\delta}{2}}{\gamma} e^{-(\Gamma + \frac{\delta}{2})x} - \frac{B}{Z_0} \frac{\Gamma + \frac{\delta}{2}}{\gamma} e^{+(\Gamma - \frac{\delta}{2})x} \\ = \frac{A}{Z_0} \frac{\Gamma - \frac{\delta}{2}}{\gamma} e^{-(\Gamma + \frac{\delta}{2})x} \left[1 - \frac{B}{A} \frac{\Gamma + \frac{\delta}{2}}{\Gamma - \frac{\delta}{2}} e^{2\Gamma x} \right], \quad (2)$$

where

$$\delta = \frac{\log_e z/z_0}{x} = \frac{\log_e y_0/y}{x} = \frac{\log_e Z/Z_0}{x} \text{ is the rate of taper,}$$

$Z_x = \sqrt{z/y} = Z_0 e^{\delta x}$ is the *surge* or *nominal characteristic impedance* of the exponential line at the point x which is equal to the characteristic impedance of the uniform line that has the same distributed constants as this line has at the point x ,

$\gamma = \sqrt{zy} = \sqrt{z_0 y_0}$ is the *propagation constant* of any uniform line that has the same distributed constants as this line at any point. It is independent of the point along the line to which it is referred, and

$\Gamma = \sqrt{\gamma^2 + \delta^2/4} = \alpha + j\beta$ is the *transfer constant* of the exponential line.

$+\gamma$ and $+\Gamma$ refer to the values of the indicated roots that are in the first quadrant.

If these equations are compared with those for a uniform transmission line it is found that the *propagation constant* is $\Gamma - \delta/2$ for voltage waves traveling in the positive x direction and $\Gamma + \delta/2$ for voltage waves traveling in the negative x direction. For current waves the corresponding *propagation constants* are $\Gamma + \delta/2$ and $\Gamma - \delta/2$. In the terminology of wave filters, Γ is the *transfer constant* and δ is the *impedance transformation constant*. $\delta/2$ is the *voltage transformation constant* and $-\delta/2$ is the *current transformation constant*. The real and imaginary parts of Γ , α and β are the *attenuation* and *phase constants* respectively.

An important parameter is

$$\nu = j \frac{\delta}{2\gamma},$$

which for a non-dissipative line is the ratio of the cutoff frequency to the frequency, as can be seen if we write the transfer constant as

$$\Gamma = \gamma \sqrt{1 - \nu^2},$$

where the indicated root is in the fourth quadrant. For a non-dissipative line ν is real and the transfer constant is real or imaginary depending on whether ν^2 is greater than or less than unity. Hence the exponential line is a high pass filter whose cutoff frequency, f_1 , is that frequency for which $\nu = \pm 1$. The transfer constant is then less than that for a uniform line by the factor $\sqrt{1 - \nu^2}$ so that both phase velocity and wave-length are larger for the exponential line than for the uniform line by the reciprocal of this factor.

If we terminate this line at $x = l$ with an impedance $Z_l = v_l/i_l$, the ratio of the reflected to direct voltage wave is found to be

$$\frac{B}{A} = - \frac{1 - (Z_l/Z_i)(\sqrt{1 - \nu^2} + j\nu)}{1 + (Z_l/Z_i)(\sqrt{1 - \nu^2} - j\nu)} e^{-2\Gamma l}, \quad (3)$$

where the coefficient of the exponential is the *voltage reflection coefficient*.

There will be no reflection if

$$Z_l = Z_i/(\sqrt{1 - \nu^2} + j\nu) = Z_l^+, \quad (4)$$

which becomes $Z_l e^{-i \sin^{-1} \nu}$ above the cutoff frequency for non-dissipative lines. This is the magnitude of the forward-looking *characteristic impedance* at $x = l$ as can be seen by dividing the first term of (1) by the first term of (2). Curve 1 of Fig. 2 gives the charac-

teristic impedance of a non-dissipative exponential line looking toward the high impedance end as a function of frequency. At infinite frequency the characteristic impedance is a resistance equal to the nominal characteristic impedance but as the frequency is decreased the phase angle of the characteristic impedance changes so that its locus

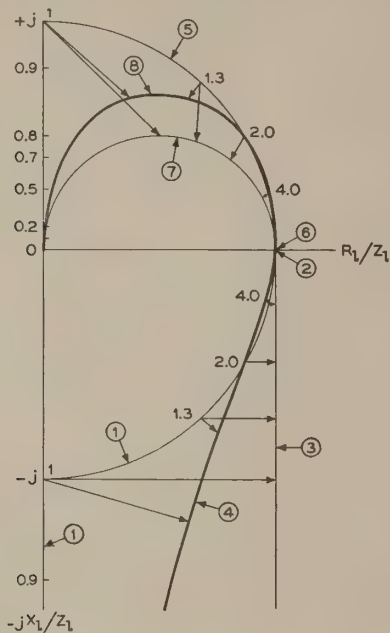


Fig. 2—Impedance diagram comparing the forward looking characteristic impedance with various terminal impedances. The numbers give the frequency relative to cutoff. The arrows are the vectors $Z_L - Z_L^+$ which are a measure of the magnitude of the reflection.

A. Step-up line.

Curve 1—Forward looking characteristic impedance,

$$Z_L^+ = Z_L e^{-j \sin^{-1}(f_1/f)}, \quad f > f_1,$$

$$Z_L^+ = Z_L [-j(f_1/f)(1 + \sqrt{1 - f^2/f_1^2})], \quad f_1 > f;$$

Curve 2—Resistance termination, $Z_L = Z_L$;

Curve 3—Capacity resistance termination, $Z_L = Z_L(1 - jf_1/f)$;

Curve 4—Capacity, resistance and inductance termination adjusted for no reflection at twice the cutoff frequency and at infinite frequency;

B. Step-down line.

Curve 5—Forward-looking characteristic impedance,

$$Z_L^+ = Z_L e^{+j \sin^{-1}(f_1/f)}, \quad f > f_1,$$

$$Z_L^+ = Z_L [+j(f_1/f)(1 - \sqrt{1 - f^2/f_1^2})], \quad f_1 > f;$$

Curve 6—Resistance termination $Z_L = Z_L$;

Curve 7—Inductance resistance termination $Z_L = Z_L/(1 - jf_1/f)$;

Curve 8—Inductance, resistance and capacity termination adjusted for no reflection at twice the cutoff frequency and at infinite frequency.

is the circular arc. At and below cutoff it is a pure reactance. If the load is a resistance equal to the nominal characteristic impedance at the terminal as indicated at 2 of Fig. 2, there will be no reflection at infinite frequency, but as the frequency is lowered there will be an increasing impedance mismatch with its accompanying reflected wave.

This reflection may be materially reduced by inserting a condenser in series with the resistance load as shown by curve 3. Further improvement results from more complicated networks. Curve 4 shows the effect of adding an inductance in shunt with the resistance load of the resistance-capacitance combination. The arrows indicate the resulting impedance mismatch which is a measure of the reflected wave.

The characteristic impedance looking toward the low impedance end is the inverse of that looking in the other direction as shown by curve 5. Shunting the resistance load with an inductance gives the impedance curve 7. Adding a capacitance element gives curve 8.

Division of (1) by (2) and substitution of the result of (3) gives the following ratio for the impedance looking into the line at the point x to the impedance level at that point,

$$\frac{Z_x}{Z_x} = \frac{K(\sqrt{1 - \nu^2} - j\nu) + 1 + [K(\sqrt{1 - \nu^2} + j\nu) - 1]e^{-2\Gamma(l-x)}}{K + j\nu + \sqrt{1 - \nu^2} - [K - \sqrt{1 - \nu^2} + j\nu]e^{-2\Gamma(l-x)}}, \quad (5)$$

where $K = Z_l/Z_l$ is the ratio of the load impedance to the impedance level at the terminal. Here as before the indicated root is in the fourth quadrant.

NETWORK CHARACTERISTICS

Three parameters are required to specify the characteristics of an exponential line of negligible loss: (1) the cutoff frequency, f_1 , (2) the length of the line which is perhaps best specified as the frequency, f_0 = velocity of light/length of line, for which the line is one wavelength long, and (3) the impedance level at some point along the line. We will designate the impedance levels at the low and high impedance ends of the line by Z_1 and Z_2 respectively, and their ratio Z_2/Z_1 by k .

When the line is terminated in a resistance equal to the impedance level at the output (5) reduces to

$$\frac{Z_1}{Z_1} = k^{\cos 2\xi} e^{2j\xi} \frac{1 + j \tan \xi k^{-\cos 2\xi}}{1 - j \tan \xi k^{\cos 2\xi}}, \quad \nu > 1, \quad (6)$$

$$\frac{Z_1}{Z_1} = e^{-2j\xi} \frac{1 + \tan \xi e^{j(2\xi + \frac{\pi}{2} - \eta)}}{1 + \tan \xi e^{j(-2\xi - \frac{\pi}{2} - \eta)}}, \quad \nu < 1. \quad (7)$$

for frequencies below and above cutoff respectively. Here $\eta = -j2\Gamma l$ is twice the electrical length of the line in radians, $\sin 2\xi = 1/\nu$, $\sin 2\xi = \nu$ and $\cos 2\xi$ is ratio of the electrical length of the line to that of a uniform line of the same physical length. For the step-down line the corresponding ratios are the reciprocal of the above expressions. These ratios are plotted in Fig. 1.

When $f \rightarrow 0$, $Z_1 = kZ_2 = Z_2 = Z_0$ and the only effect of the line is to connect the load to the input. Above cutoff the magnitude of the input impedance oscillates about the nominal characteristic impedance and the phase angle oscillates about the value $-2\xi (\approx -f_1/f$ for $f \gg f_1$) which goes from $-\pi/2$ to 0 as the frequency increases indefi-

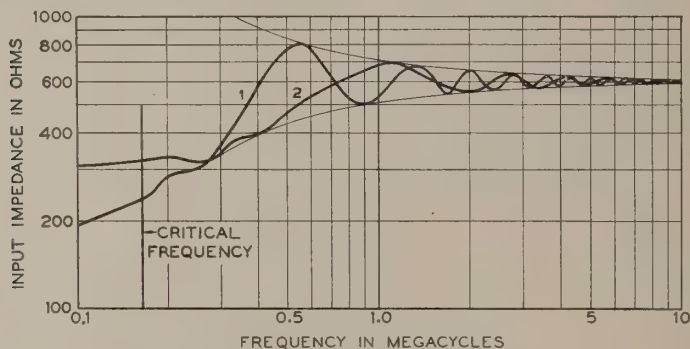


Fig. 3—Input impedance characteristics.
Curve 1—150 : 600 ohm line, 100 meters long.
Curve 2—300 : 600 ohm line, 200 meters long.
Both lines have the same rate of taper.

nately from cutoff. The variation of the input impedance with frequency is shown for two lines of different length but the same rate of taper in Fig. 3. The magnitude of the oscillations depends only on the rate of taper and decreases with increase in frequency. The impedance varies between $(1 + f_1/f)$ and $1/(1 + f_1/f)$. The positions of the maxima and minima, however, are determined by the length of the line. They occur respectively at those frequencies for which the line is approximately $1/8$ of a wave-length more than an even or an odd number of quarter wave-lengths long. The phase angle is usually negative but has a small positive value when the line is approximately a half wave-length long.

The locations of these maxima and minima are the same as would result from terminating a uniform line in an impedance whose magnitude is the same as the characteristic impedance but has a small reactive component. This suggests adding a compensating reactance

to the resistance load. From (3) the best single reactive element is found to be a condenser whose impedance is equal to the impedance level at the cutoff frequency. This gives a value of $K = 1 - j\nu$ which when substituted in (5) shows that the input impedance is to a first approximation a constant times the terminal impedance. To correct for the reactive component of the input impedance an inductance having an impedance jZ_1/ν which is equal to the input impedance level at cutoff is shunted across the input. The resulting impedance transforming network consists of an exponential line with a series capacitance at the high impedance end and a shunt inductance at the low impedance end. When terminated in a resistance load at either end equal to the impedance level at that end the input impedance, to a first approximation, is a resistance equal to the impedance level at the input end. In fact the deviations of the input impedance from the ideal for transmission in one direction are just the reciprocal of those for transmission in the other direction.

The magnitudes of the series capacitance and shunt inductance that give the improved network may be expressed in terms of the electrostatic capacitance and loop inductance of the line. Simple calculation shows that the required series capacitance is equal to $2/(k - 1)$ times the electrostatic capacitance of the line and the required shunt inductance is equal to the same factor times the total inductance of the line.

There is an interesting relationship between these terminations and a simple high-pass filter. The LC product of the shunt and series arms of the filter resonates at f_1 . If an ideal transformer with transformation ratio k is inserted between the shunt inductance and the series capacitance, the capacitance becomes C/k and the new LC resonates at $f_1\sqrt{k}$. This is the same frequency at which the series capacitance and shunt inductance that are added to the terminations of the exponential line resonate. Furthermore the reactance of the shunt inductance is equal to the impedance level at the cutoff frequency and the reactance of the series capacitance is equal to the impedance level at the cutoff frequency exactly as in the case of the high-pass filter.

By using the exponential line it is possible to construct a network with properties that no network with lumped circuit elements possesses, namely, a high-pass impedance transforming filter.

CRITICAL LENGTHS

Besides the characteristics of the exponential line that are substantially independent of the length of the line, it has properties that

depend on the length of the line that are analogous to those of a uniform line a half wave-length or quarter wave-length long. For non-dissipative lines above the cutoff frequency (5) becomes

$$Z_1 = \frac{K \cos\left(\frac{\eta}{2} - 2\xi\right) + j \sin \frac{\eta}{2}}{\cos\left(\frac{\eta}{2} + 2\xi\right) + jK \sin \frac{\eta}{2}} Z_1. \quad (8)$$

When the line is an integral number of half wave-lengths long ($2\eta = \pi$) this reduces to

$$Z_1 = KZ_1 = kZ_2, \quad (9)$$

which says that the input impedance is equal to the impedance transformation ratio times the load impedance. The length of exponential line that corresponds to a quarter-wave uniform line differs from an odd multiple of a quarter wave-length by an amount such that

$$\tan\left(\frac{\eta - (2n + 1)\pi}{2}\right) = \frac{K^2 - 1}{K^2 + 1} \tan 2\xi, \quad (10)$$

for which (8) becomes

$$Z_1 = \frac{Z_1 Z_2}{Z_2}. \quad (11)$$

Similar expressions exist for the step-down line, but $1/K$ must be substituted for K in (10) for the length corresponding to the quarter-wave uniform line.

WITH DISSIPATION

An exponential line is an improvement over the uniform "iron wire" line as a resistance load that will dissipate a large amount of power.

Provided the attenuation is not too large the current and voltage distribution will be the same as for a non-dissipative line except for the additional power loss so that we may use the equations for an exponential line even though the distributed series resistance and shunt leakage do not vary exponentially with distance.

Suppose that the conductor size and resistance that will just dissipate the desired input power result in an attenuation constant α_0 for a uniform transmission line. To a first approximation the conductors can carry the same current irrespective of the impedance level. The current wave will be given by the first term of equation (2) which becomes

$$i = i_0 e^{-(\delta/2)x - \alpha_0 x},$$

except for a phase factor. In order that the current will not increase, $\delta = -2\alpha$. The actual attenuation "constant,"

$$\alpha_x \sim \left(\frac{R}{2Z_x} + \frac{GZ_x}{2} \right) \left(1 + \frac{1}{2} \frac{f_1^2}{f^2} \right) \sim \alpha_0 e^{2\alpha_0 x} \left(1 + \frac{1}{2} \frac{f_1^2}{f^2} \right), \quad (13)$$

will increase with distance down the line so that the current will decrease but not as rapidly as with a uniform line. The total attenuation in nepers is approximately

$$\left(1 + \frac{1}{2} \frac{f^2}{f_1^2} \right) \int_{x=0}^l \alpha_0 e^{2\alpha_0 x} dx = \left(1 + \frac{1}{2} \frac{f^2}{f_1^2} \right) \left(\frac{e^{2\alpha_0 l} - 1}{2} \right). \quad (14)$$

At the point where the attenuation of the uniform line is 6 db the tapered line has an additional attenuation of 7 db above the uniform line or a total attenuation of more than twice. The current has been reduced to less than half. Here an improvement may be made by increasing the dissipation by either changing the wire size or resistivity of the conductor. A greater improvement would result from changing the resistivity because then the capacity for heat dissipation would be the same. Suppose, however, that one conductor material is to be used throughout and the dissipation capacity is proportional to the wire surface; then at this point the wire size could be reduced to 1/2, doubling the attenuation factor. It is already 4 times that for the uniform line, so this increases it to 8 times. The resulting total attenuation is 30 db in a length that would have less than 7 db if the line were uniform. If this attenuation were required the length of line could be reduced by a factor of about 4.4. Of course the spacing is very close at the end of this line, but the line could be shorted at the end. This would approximately double the current at the end, but here again the current carrying capacity of the line is more than double the current traveling down the line. With the line shorted the reflected current would be 60 db down, which would not affect the input impedance appreciably. For the first 13 db of attenuation the impedance of the line would be relatively free from changes due to changes in spacing resulting from wind, etc. When the spacing is small enough to be affected by wind, vibration, etc., the attenuation will be great enough to suppress these small irregularities.

EXPERIMENT

In order to verify the foregoing theoretical development, measurements have been made on several experimental lines. Figure 4 shows the results of measurements on two such lines. These lines were

constructed of No. 12 tinned copper. At the low impedance end the strain was taken by a victron insulator which also served as a line spreader and terminal mounting. At the high impedance end the strain was taken by 1/4" manila rope without other insulation. The line spacing was adjusted by "lock stitch" tension insulators spaced 1 meter and 1/2 meter apart on the low and high impedance end respectively of the 9-meter line. The 3-meter line was supported at the 1/4, 1/2, 2/3, 3/4 and 7/8th points.

The impedance was measured by the substitution method. To facilitate the substitution of the reactive component of the line it was bridged by an antiresonant circuit. Pencil leads calibrated on direct current were used as the resistance standards. Type BW IRC 1/2

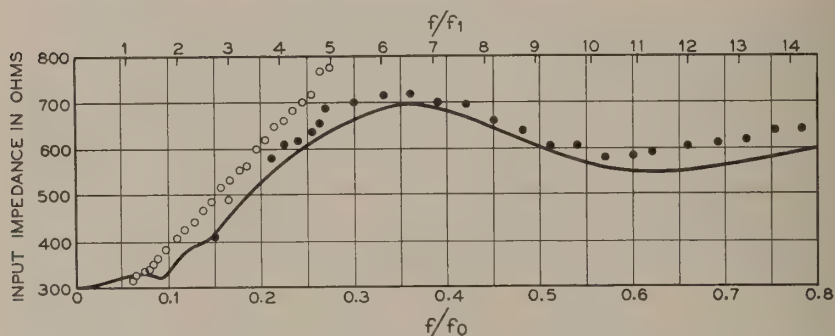


Fig. 4—Input impedance characteristic. Comparison of theoretical curve with experimental points for 600 : 300 ohm lines.

Solid circles—9-meter line.
Open circles—3-meter line.

watt resistances were used for terminations. The solid circles of Fig. 4 represent measurements on the 9-meter line. The agreement with theory is as good as is usually found for actual "uniform lines." In order to check the theory further toward the lower frequency end—beyond the range of the measuring equipment—measurements were made on a 3-meter line. These measurements are shown by the open circles. The agreement with theory is not so good, but here the lengths of the connecting leads are an appreciable fraction of the length of the line.

Preliminary tests on a full size model of exponential line impedance transformer showed deviations from the theoretical that might be attributed to improper termination, irregularities along the line, irregularities introduced at the change in conductor size or capacitance of the spacing insulators. Since it was impossible to determine which of these was the predominant cause of the deviations from the ideal, it

was decided to introduce each of these factors one at a time. This test was made on a 600 : 300 ohm line constructed of No. 6 copper wire with lockstitch insulators except at the terminals. The correct termination was obtained by tests on a uniform 300 ohm line with the same physical structure at the termination. Of necessity the tying of the wire to the strain insulators at the end introduced a shunt capacitance which augmented the inherent additional capacitance due the "end effect." This additional capacitance is equal to that of a short length of line.

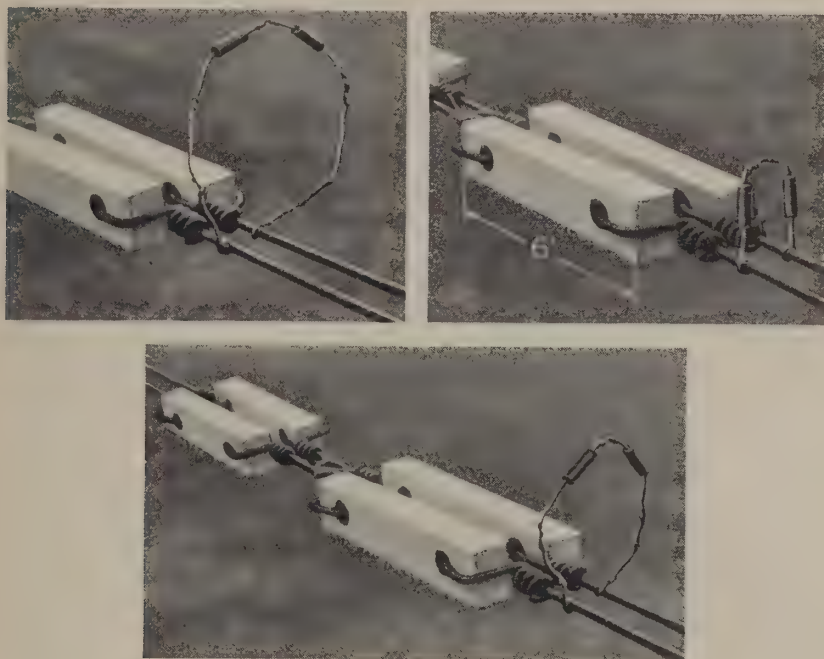


Fig. 5—Photographs of terminations of 300 ohm line.
 upper right for curve 1
 lower for curve 2
 upper left for curve 3 } of Fig. 6.

If the correct amount of inductance is inserted in series with the resistance load the combined effect of the additional capacitance and inductance becomes the same as the addition of a small length of line for all frequencies up to those for which this length of line is an appreciable fraction of a wave-length. Accordingly a small amount of inductance was inserted in series with the resistance as shown in the right picture of Fig. 5. The input impedance of the uniform line with this termination is given by "Experimental Curve 1" at the bottom of Fig. 6.

A three-inch length of No. 18 wire was inserted as shown in the lower picture of Fig. 5 and "Experimental Curve 2" resulted. This reduced the irregularities in the input impedance to about half, so another three

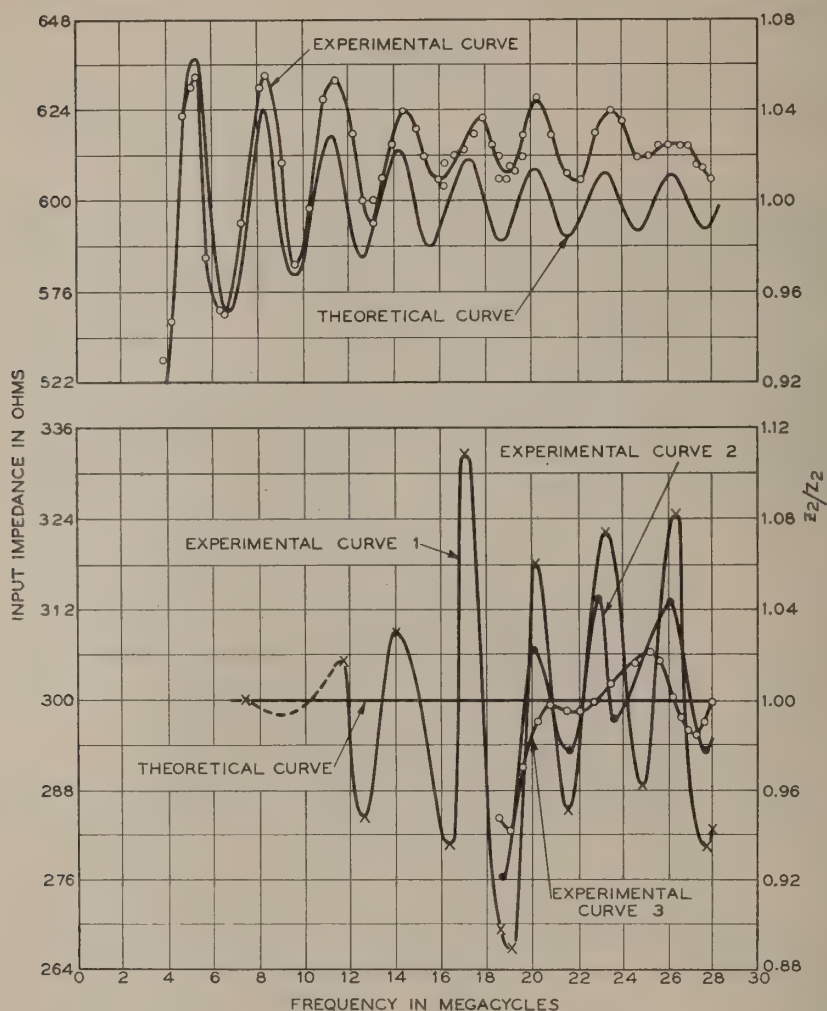


Fig. 6—Lower. Experimental input impedance characteristics of 300 ohm line with terminations shown in Fig. 5. Upper. Input impedance characteristics of 50-meter 600 : 300 ohm line of No. 6 conductors.

inches were inserted, resulting in "Experimental Curve 3." Here the maxima and minima are displaced, indicating that the effect of the stray capacitance has been reduced to the same order of magnitude as that due to the deviation of the resistance from the desired value. This

termination was accordingly removed to the exponential line, resulting in the "Experimental Curve" at the top of Fig. 6. It agrees within experimental error with the "Theoretical Curve." The slight vertical displacement of the experimental curve at the higher frequencies is attributed to deviations in the impedance of the pencil lead, which was

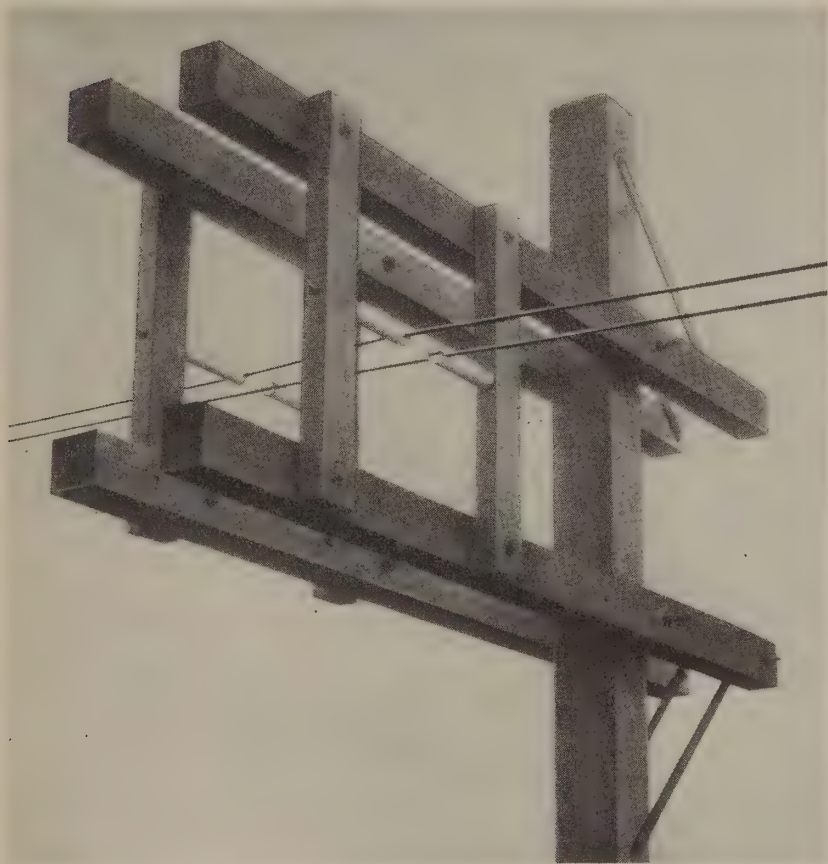


Fig. 7—Photograph of one of the changes in conductor size.

used as a resistance standard, from a pure resistance equal to its direct current value.

To increase the power carrying capacity of the exponential line, one was built with larger wire size at the lower impedance end. This increased the breakdown voltage by increasing the spacing and conductor diameter and at the same time increased the current carrying capacity by decreasing the resistance and increasing the heat dissipat-

ing capacity of the conductors. This was a 600 : 300 ohm line constructed of 20 meters No. 6 wire, 10 meters 1/4" tubing and 20 meters 3/8" tubing. Here again the correct termination was determined by measurements on a 300 ohm uniform line of 3/8" tubing. The total length of terminating loop that gave the best termination was $6\frac{1}{2}$ " in this case compared with $10\frac{1}{2}$ " for the 300 ohm line of No. 6 wire. Since no attempt was made to reduce the variations in input impedance to less than ± 1 per cent these lengths may be as much as an inch off.

These measurements indicated that the exponential line would perform satisfactorily as an impedance transformer if it could be constructed to have the desired mechanical features without impairing its electrical properties. The greatest difficulty appeared to reside in the

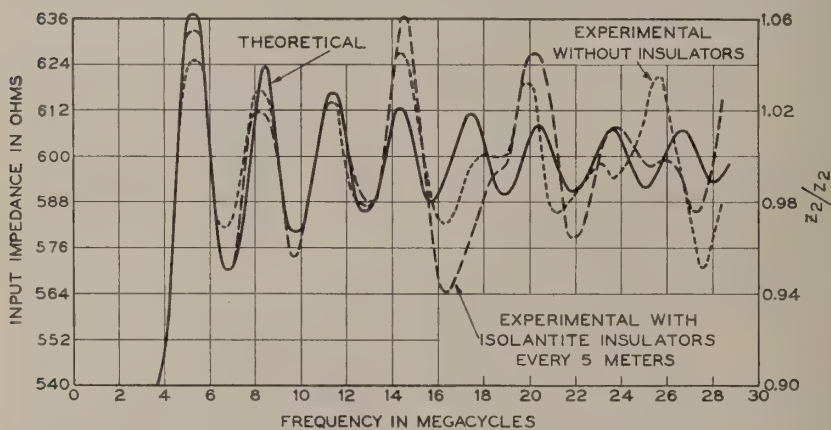


Fig. 8—Input impedance characteristics of 50-meter 600 : 300 ohm line of 3/8", 1/4" and No. 6 conductors.

insulators. Special isolantite insulators were designed that would be satisfactory commercially and still keep the additional capacity to a reasonable value. Figure 7 shows the construction of the line at the supporting poles where the conductor size changes.

The results of measurements on this line are shown in Fig. 8. The solid curve was calculated from the equations developed earlier. The two broken curves are the results of measurements on the line, one without insulators and one with insulators. While the insulators affect the line somewhat they do not increase the deviation from the ideal appreciably. [The improvement in the agreement between experiment and theory in this set of curves over that in Fig. 4 is presumably due to the fact that the comparison resistance for Fig. 8 consisted of 3-IRC

resistances instead of the pencil lead. With the fixed IRC resistance it was, of course, impossible to adjust the standard to exactly the same value as the unknown. In this case the small difference was determined by using the slope of the rectifier voltmeter calibration.] This line has a maximum deviation from the desired input impedance of ± 6 percent for all frequencies above 4.2 mc. (Measurements were made up to 28 mc.) The phase angle of the input impedance was found to be zero

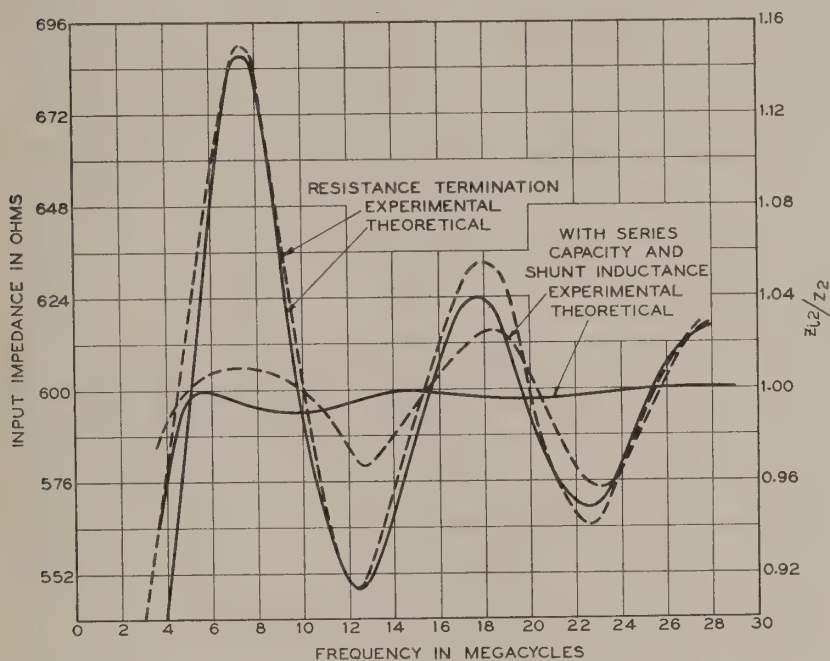


Fig. 9—Input impedance characteristics of 15-meter 600 : 300 ohm line of No. 6 conductors

within the accuracy of measurement. From theory the phase angle would be expected to vary between -0° and $+3^\circ$.

The curves of Fig. 9 refer to a 600 : 300 ohm line of No. 6 wire 15 meters long. With resistance termination this line has rather large variations in the input impedance but with the addition of the proper reactances the input impedance is flatter than the longer line with resistance termination. At the lower frequencies where the variations in the input impedance were large without the reactive networks, their addition gives approximately the expected improvement. At the higher frequencies the inductance was approximately anti-resonated

by its distributed capacity and the input impedance approaches that for the resistance termination.

CONCLUSION

Theory indicates that the exponential line may be used as an impedance transformer over a wide frequency range. The results of experiment show that the desired characteristic can be realized in practice. Among the applications of the exponential line may be mentioned its use in transforming the impedance level back to its original value after the paralleling of two transmission lines feeding two antennas. It could be used to transform the input impedance of a rhombic antenna down to the usual 600-ohm level of open wire transmission lines. If twin coaxial lines are used inside the transmitter building to eliminate undesired feedback, coupling, etc., the exponential line could be used to transform from the highest practical impedance level of such lines to a practical level of the more economical open wire lines for use outside the building.

APPENDIX

The exponential line is a non-uniform line so that the terms "characteristic impedance" and "surge impedance" of an exponential line are not synonymous. The terms "surge impedance"¹ and "nominal characteristic impedance"² may be used synonymously for the characteristic impedance of the uniform line that has the same distributed constants as the non-uniform line at the point in question. Expressed as functions of the distributed "constants" of the line they are the square root of the ratio of the distributed series impedance to the distributed shunt admittance at the point along the line in question. It will be expedient to refer to the nominal characteristic impedance as the impedance level at the point in question. Schelkunoff³ has defined the characteristic impedances as the ratio of voltage to current at the point in question for each of the two traveling waves of which

¹ The term "surge impedance" is defined by A. E. Kennelly on page 73 of "The Applications of Hyperbolic Functions to Electrical Engineering Problems" (McGraw-Hill 1916) as follows: "The surge impedance of the line is not only the natural impedance which it offers everywhere to surges of the frequency considered, but it is also the initial impedance of the line at the sending end." Hence the "surge impedance" should be independent of the configuration of the line except at the point in question and in particular it should be equal to that for a uniform line constructed so as to have the same dimensions everywhere as the non-uniform line has at the point in question.

² The word nominal as used here has the same meaning as in "nominal iterative impedance" as used by K. S. Johnson in "transmission circuits for telephone communication" (Van Nostrand 1925).

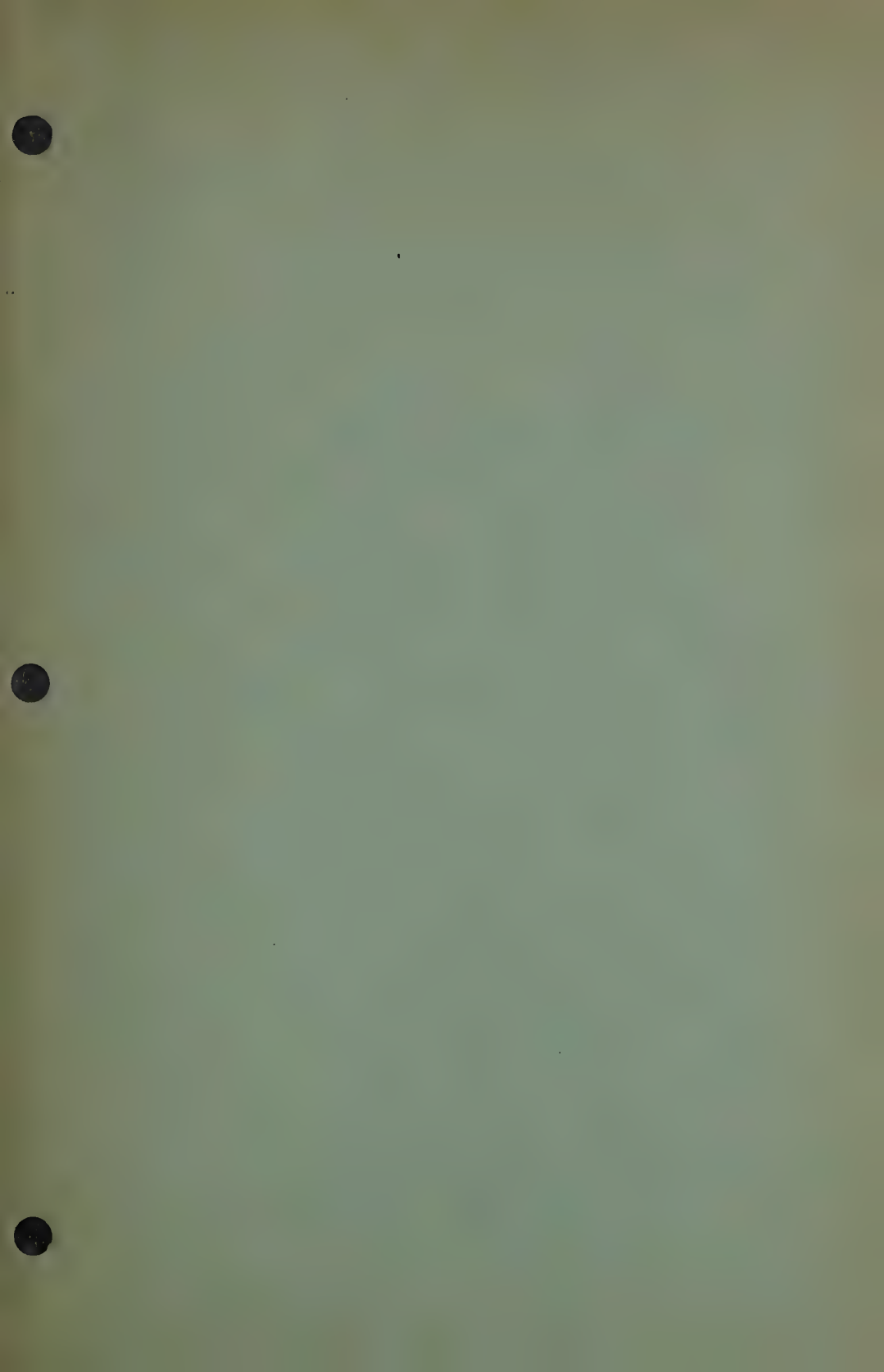
³ S. A. Schelkunoff, "The Impedance Concept and its Application to Problems of Reflection, Refraction, Shielding and Power Absorption," *Bell System Technical Journal*, 17, 17-48, January, 1938.

the steady state condition is composed. At each point an exponential line has two characteristic impedances which are different and depend upon the frequency as well as the position along the line.

Because of the change of impedance level, the propagation constants for the voltage and current differ, so that it is convenient to consider the transfer constant ⁴ which may be defined as half the sum of the voltage and current propagation constants.

⁴ Compare with the definition of "image transfer constant" as given by K. S. Johnson in "Transmission Circuits for Telephone Communication."





BELL TELEPHONE LABORATORIES
INCORPORATED
463 WEST STREET, NEW YORK

Transmission Line Theory. (General)

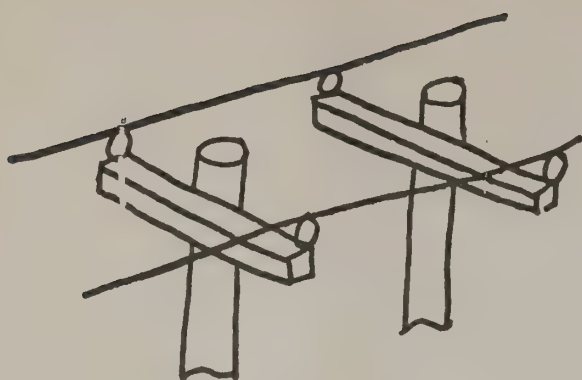
1. Introductory

The transmission lines treated in these notes consists of two conductors, usually copper or aluminum, arranged in various configurations some of which are illustrated symbolically by Fig.1. The purpose of a transmission line is to guide electrical energy in some form, such as 60 cycle energy for home and commercial purposes or modulated energy in the form of telephone conversation and others, from one location to another. The conductors of the transmission line do not transmit the energy but serve only as guides for the energy which is transmitted in the electromagnetic field surrounding the conductors. This is the only concept that satisfies the Maxwell field equations and agrees with other systems of energy transmission such as hollow wave guides where only one conductor exists and radio transmission where there are no conductors.

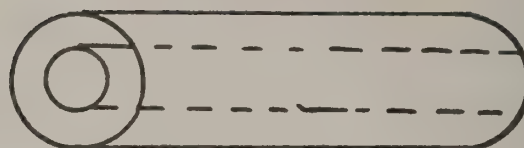
The mathematical treatment and physical behavior of a transmission line depends upon whether the line is electrically short or long. Actual physical length and electrical length are two different quantities when deciding whether a line is short or long. A transmission line is 100 meters long physically. At 60 cycles per second it is very short electrically. At 10^8 cycles per second it is very long electrically. Electrical length is expressed in terms of wavelength for a particular operating frequency. Wave length is the distance for which a voltage, or current, undergoes a 360° phase shift from one end of the line to the other. One wave length at 60 cps is 3100 miles. Whereas it is only 3 meters at 10^8 cps.

A transmission line is a linear circuit consisting of resistance R, inductance L, capacitance C and leakage conductance G uniformly distributed throughout the length of the line. The resistance and inductance are in series with the line conductors and the capacitance and leakage conductance are in shunt as illustrated in Fig. 2. Because of the distributed nature of these parameters the current and voltage on a line undergo continual changes in magnitude and phase along the line in relation to the current and voltage at some reference point such as the sending end or the receiving end.

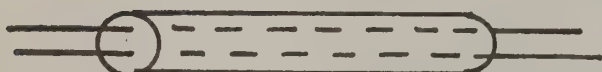
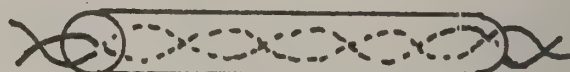
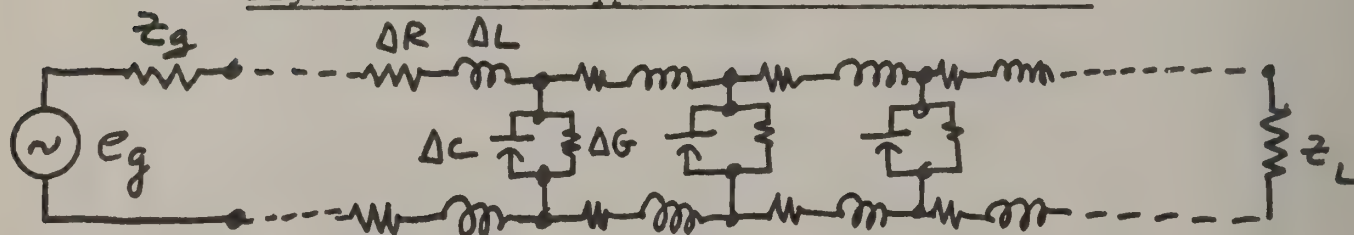
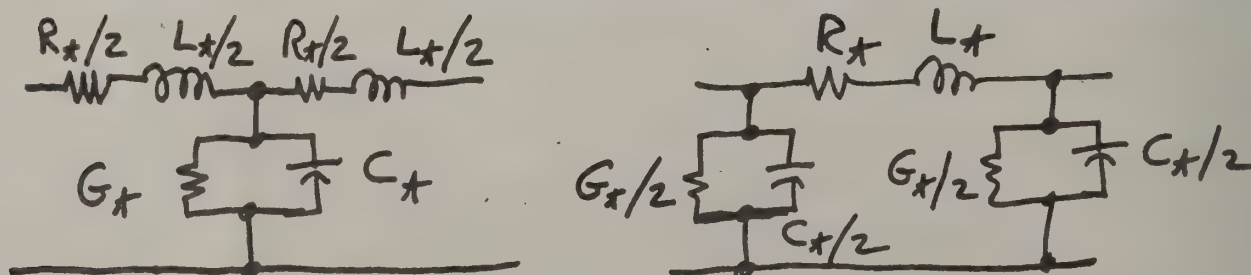
A very short, electrically, length of transmission line may be represented as a T or π network as shown in Fig. 3. For example 50 miles of 60 cycle power line is only .0155 wave lengths long and may therefore be represented as a single T or π network. For a power line only a few miles long it is generally sufficient



a. Open wire line



b. Coaxial line

c. Shielded pair line.
Outside shield usually
copper and grounded.d. Cable type line. Sheath may be
metallic. Some cables contain
several pairs of conductors.Fig. 1. Several Types of Transmission LinesFig. 2. A very short piece of transmission line between the
source e_g and the load Z_L , showing how R , L , C and G are
disposed to form a circuit of uniformly distributed parameters.Fig. 3. A very short transmission line represented by a T or π
network. The parameters R_t , L_t , C_t and G_t are for the entire
length of line.

to replace the line by its overall resistance and inductance and neglect its capacitance and leakage conductance.

When a line becomes of the order of .05 wave lengths or longer and there is only one frequency of interest the line may be represented by a single T or π network whose parameters are determined from the open and short circuit impedances of the line. However, open and short circuit impedances must be determined either from measurements or from mathematical theory which takes into consideration the uniform distribution of the line parameters. Thus for a line that cannot be represented by a T or π network without the use of the open and short circuit impedance it becomes simpler to treat the line as a uniformly distributed parameter circuit and derive and solve the differential equation which apply.

2. Derivation of the Differential Equations for the Transmission Line.

In order to arrive at a pair of equations which express the relations for voltage and current at any point on a transmission line it is necessary to first derive the fundamental differential equations for the line. These equations are then solved for voltage and current in terms of the line parameters, the distance to a point in question and the terminal conditions.

The differential equations are developed by applying the e.m.f. and current laws to an infinitesimal portion of the line. Fig. 4 represents an infinitesimal portion of a transmission line. Since the current in one conductor is equal to, but opposite in direction, at any instant of time to the current in the other conductor, all line resistance and inductance of the infinitesimal portion of line is placed in the upper branch as shown.

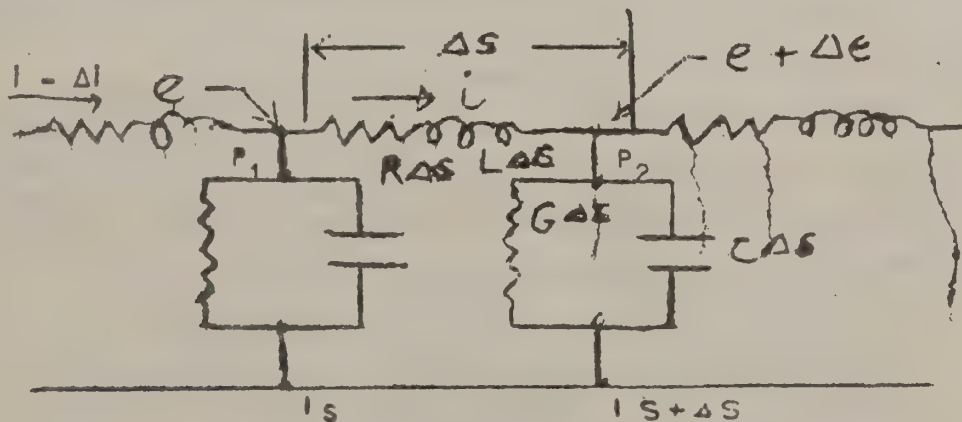


Fig. 4 An infinitesimal length of transmission line.
The parameters R, L, C and G are for a unit length of line.

Referring to Fig. 4 the difference in potential between the line conductors at point P_2 is equal to the potential difference e at P_1 plus Δe where Δe is the incremental change in potential difference in going from P_1 to P_2 because of the current i in RAs and LAs. In equation form this becomes

$$e - (Ri + L \frac{\partial i}{\partial t}) \Delta s = e + \Delta e$$

Which gives, as $\Delta s \rightarrow 0$

$$\frac{\partial e}{\partial s} = - (Ri + L \frac{\partial i}{\partial t}) \quad (1)$$

In the infinitesimal portion of line from P_1 to P_2 the current i and potential e are changing with respect to both time and distance, thus the partial derivatives.

In a similar manner as the above the current to the right of P_2 differs from the current between P_1 and P_2 by Δi where Δi is the incremental change in current due to the shunt paths GAs and CAs. In equation form

$$i \approx \Delta i - \Delta i - (Ge + C \frac{\partial e}{\partial t}) \Delta s$$

Which gives, as $\Delta s \rightarrow 0$

$$\frac{\partial i}{\partial s} = - (Ge + C \frac{\partial e}{\partial t}) \quad (2)$$

Equations 1 and 2 are the fundamental differential equations for a transmission line. The equations give the space rate-of-change of e and i respectively, and at each point on the line e and i are changing with respect to time. This indicates that e and i are propagated along the line in a wave pattern. This will become clearer when the differential equations are solved and physical interpretations are given for the resulting solutions.

Solutions of equations 1 and 2 are best carried out by first obtaining two equations, each of which is expressed in terms of a single dependent variable, i.e. one equation containing only e and the other containing only i . This ^{is} carried out as follows.

Differentiation of 1 with respect to s gives

$$\frac{\partial^2 e}{\partial s^2} = - (R \frac{\partial i}{\partial s} + L \frac{\partial^2 i}{\partial t \partial s}) \quad (3)$$

Differentiation of 2 with respect to t gives

$$\frac{\partial^2 i}{\partial t \partial s} = (G \frac{\partial e}{\partial t} + C \frac{\partial^2 e}{\partial t^2}) \quad (4)$$

Now substitute $\frac{\partial i}{\partial s}$ from equation 2 and $\frac{\partial^2 i}{\partial t \partial s}$ from equation 4 into equations 3 and get

$$\frac{\partial^2 e}{\partial s^2} = RGe + (RC + LG) \frac{\partial e}{\partial t} + LC \frac{\partial^2 e}{\partial t^2} \quad (5)$$

In like manner

$$\frac{\partial^2 i}{\partial s^2} = RGi + (RC + LG) \frac{\partial i}{\partial t} + LC \frac{\partial^2 i}{\partial t^2} \quad (6)$$

These are the second order differential equations that govern the behavior of e and i for the transmission line.

Except for the special case which is the steady state solution when e and i are sinusoidal functions of time the solutions of equations 5 and 6 are quite complicated both mathematically and physically. Hence in order to acquire some physical insight into the nature of e and i before proceeding with steady state sinusoidal solutions equation 5 and 6 will be modified to represent the loss-less line condition, $i, e, R = G = 0$. In this sense they become

$$\frac{\partial^2 e}{\partial s^2} = LC \frac{\partial^2 e}{\partial t^2} \quad (7)$$

$$\frac{\partial^2 i}{\partial s^2} = LC \frac{\partial^2 i}{\partial t^2} \quad (8)$$

3. Travelling Waves.

Equations 7 and 8 are generally known as the wave equations for loss-less transmission lines. Just why these equations portray wave motion may be seen by examining in some detail their steady state solutions when e and i are sinusoidal functions of time. An assumed solution for equation 7 is

$$e = e_1 \cos (wt - \beta s) + e_2 \cos (wt + \beta s) \quad (9)$$

That this is a solution may be shown as follows:

$$\frac{\partial e}{\partial s} = + \beta e_1 \sin (\omega t - \beta s) - \beta e_2 \sin (\omega t + \beta s)$$

$$\begin{aligned} \frac{\partial^2 e}{\partial s^2} &= - \beta^2 e_1 \cos (\omega t - \beta s) - \beta^2 e_2 \cos (\omega t + \beta s) \\ &= \beta^2 [-e_1 \cos (\omega t - \beta s) - e_2 \cos (\omega t + \beta s)] \end{aligned}$$

Likewise

$$\begin{aligned} \frac{\partial^2 e}{\partial t^2} &= - \omega^2 e_1 \cos (\omega t - \beta s) - \omega^2 e_2 \cos (\omega t + \beta s) \\ &= \omega^2 [-e_1 \cos (\omega t - \beta s) - e_2 \cos (\omega t + \beta s)] \end{aligned}$$

Then if $\beta = \omega \sqrt{LC}$ equation 9 becomes a solution of equation 7. Now by graphing the first term of equation 9 in Fig. 5 for several instants of time it is seen that this term represents a wave moving in the positive direction of s . In a similar manner the second term of equation 9 represents a wave moving in the negative s direction. For one complete period, i.e. $1/\text{frequency}$, the wave moves a distance called one wave length λ . Thus

$$s_\lambda = \lambda = \frac{2\pi}{\beta} \quad \text{OR} \quad \beta = \frac{2\pi}{\lambda} \quad (10)$$

Now s = velocity of wave multiplied by time or

$$s_\lambda = \lambda = \text{vel.} \cdot t = \text{vel.} / f = \frac{2\pi}{\beta}$$

$$\text{Hence} \quad \text{vel} = \frac{2\pi f}{\beta} \quad (11)$$

$$\text{and} \quad \beta = 2\pi f \sqrt{LC} \quad (12)$$

Consequently velocity = $\frac{1}{\sqrt{LC}} = v_p$ and is called the phase velocity of the wave. For a transmission line with air or vacuum dielectric it may be shown that if the self inductance due to magnetic flux linkages inside the conductors is neglected $1/\sqrt{LC} = c$ the velocity of light in free space. For any lossless line $v_p = c/\sqrt{\epsilon_r}$ where ϵ_r is the dielectric constant of the medium around the line conductors.

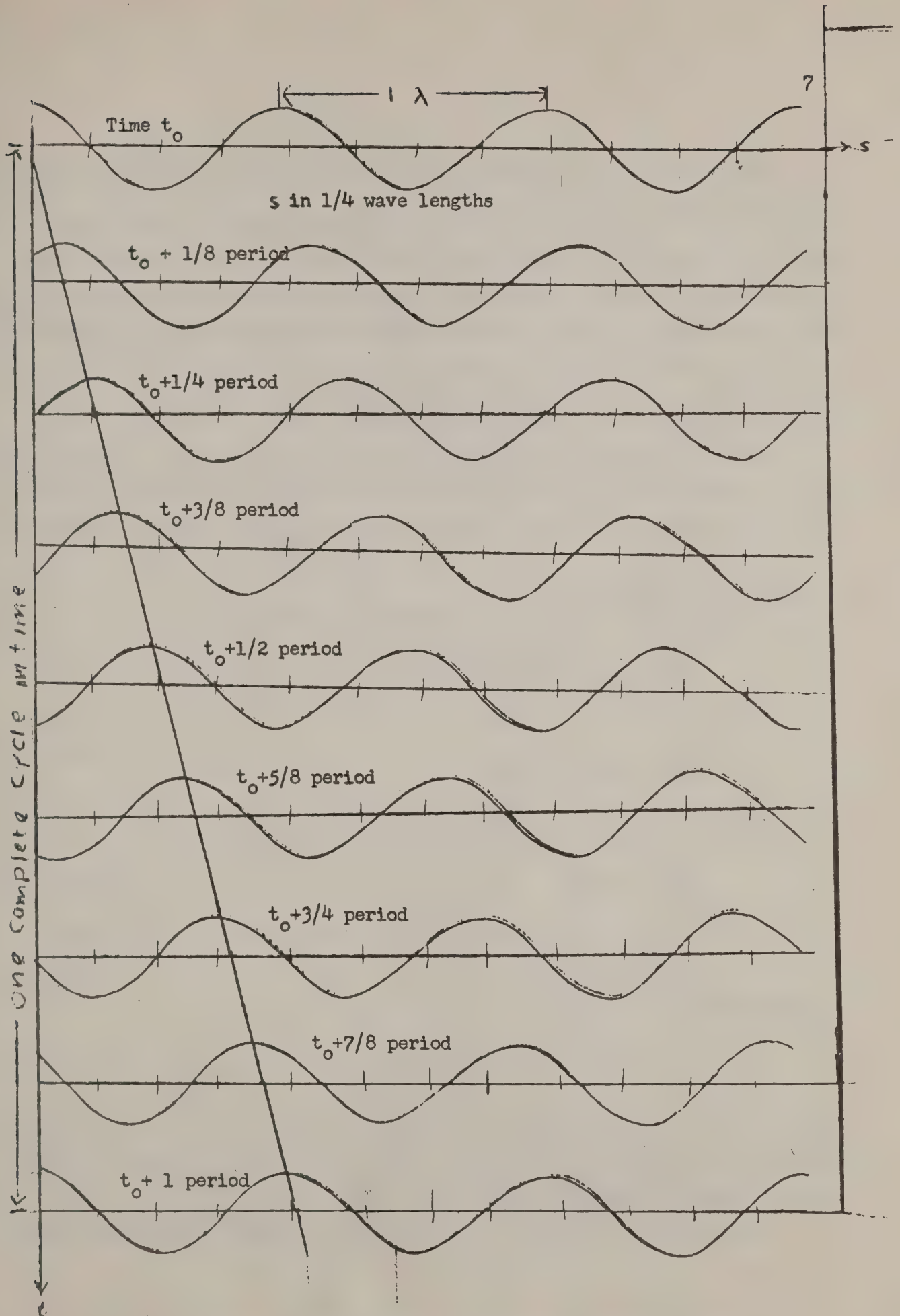


Fig. 5. Wave moves to right 1λ in 1 cycle of time

Assume now a line so terminated that only the first term of equation 9 exists. This is possible as will be seen later. Suppose two oscilloscopes were connected across the line s meters apart. Both oscilloscopes would show sinusoidal voltage - time waves. However, the voltage-time wave shown on the oscilloscope farthest from the source end would be βs degrees lagging the voltage-time wave shown by the other oscilloscope. Suppose the oscilloscope nearest the source were connected to the line at the instant the voltage was e_a volts and then moved along the line at a velocity v_p . The oscilloscope would continue to show e_a volts. These observations help to show that a travelling wave exists on the line.

When the resistance and leakage conductance are not zero, as is the case for a realizable line, the travelling waves suffer attenuation along the line. This will be shown later when the equations for the general case are discussed.

4. Transient considerations.

Returning now to the basic differential equations namely

$$\frac{\partial e}{\partial s} = -(Ri + L \frac{\partial i}{\partial t})$$

$$\frac{\partial i}{\partial s} = -(Ge + C \frac{\partial e}{\partial t})$$

and setting $R = G = 0$ for the loss less case there results

$$\frac{\partial e}{\partial s} = -L \frac{\partial i}{\partial t} \quad (13)$$

$$\frac{\partial i}{\partial s} = -C \frac{\partial e}{\partial t} \quad (14)$$

A general solution of equation 13 for e is

$$e = e_1 f(t - \frac{s}{v}) + e_2 f(t + \frac{s}{v}) \quad (15)$$

where $v = 1/\sqrt{LC}$ and is the velocity of propagation as was found for the steady state solution when e and i vary sinusoidally with time. It may be shown from fundamental dimension that $1/\sqrt{LC}$ is velocity.

Now examining, in detail the first term of equation 15 it is seen that

$$\begin{aligned}\frac{\partial e}{\partial s} &= e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \frac{\partial(t - \frac{s}{v})}{\partial s} \\ &= e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \left(-\frac{1}{v}\right)\end{aligned}\quad (16)$$

Assume now the corresponding solution for i , i.e.

$$i = \frac{e_1}{R_o} f(t - \frac{s}{v}) - \frac{e_2}{R_o} f(t + \frac{s}{v}) \quad (17)$$

and taking the partial derivative of the first term, i.e.

$$\frac{\partial i}{\partial t} = \frac{e_1}{R_o} \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})}$$

Then

$$e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \left(-\frac{1}{v}\right) = -L \frac{e_1}{R_o} \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \quad (18)$$

$$\text{This is true if } R_o = \sqrt{\frac{L}{C}} \quad (19)$$

The $\sqrt{\frac{L}{C}}$ is called the characteristic resistance of the loss-less line and is designated as R_o or R_c .

Proceeding in a similar manner for ^{the} second terms of e and i ^{again} results in

$$R_o = \sqrt{\frac{L}{C}}.$$

$$\text{Then } i = \frac{e_1}{\sqrt{L/C}} f(t - \frac{s}{v}) - \frac{e_2}{\sqrt{L/C}} f(t + \frac{s}{v}) \quad (20)$$

Now inasmuch as $e_1 f(t - \frac{s}{v})$ is a wave travelling in the positive direction of s and $e_2 f(t + \frac{s}{v})$ is travelling in the negative s direction it is proper to call $e_1 f(t - \frac{s}{v})$ an incident wave and $e_2 f(t + \frac{s}{v})$ a reflected wave.

Hence

$$e = e_1 + e_r \quad \text{where} \quad (21)$$

$$e_1 = e_1 f(t - \frac{s}{v}) \quad \text{and} \quad e_r = e_2 f(t + \frac{s}{v})$$

Likewise

$$i = i_1 + i_r \quad \text{where} \quad (22)$$

$$i_1 = \frac{e_1}{R_o} f(t - \frac{s}{v}) \quad \text{and} \quad i_r = -\frac{e_2}{R_o} f(t + \frac{s}{v})$$

$$\text{where } R_o = \sqrt{L/C}$$

Suppose now a loss-less line is terminated in a resistance R_L , then $e_L = i_L R_L$. Hence

$$\begin{aligned} e_L &= e_{1L} + e_{rL} \\ i_L &= \frac{e_{1L}}{R_o} - \frac{e_{rL}}{R_o} \end{aligned} \quad (23)$$

The solution of these two equations yields

$$\frac{e_{rL}}{e_{1L}} = \frac{R_L - R_o}{R_L + R_o} = K_{eL} \quad (24)$$

$$\frac{i_{rL}}{i_{1L}} = -\frac{R_L - R_o}{R_L + R_o} = K_{iL} = -K_{eL}$$

Equations 24 give the voltage reflection coefficient K_e and current reflection coefficient K_i both at ^{the} load resistance R_L . In other words

$$e_{rL} = K_{eL} e_{1L} \quad \text{and} \quad i_{rL} = K_{iL} i_{1L}$$

For the case in which $R_L = R_0$, $K_{eL} = K_{iL} = 0$ and there is no reflection. That is, all of the energy that reaches the load is dissipated in the load.

When the line is shorted at the load $R_L = 0$, $K_{eL} = -1$ and $K_{iL} = +1$. When the line is open at the load $R_L = \infty$, $K_{eL} = +1$ and $K_{iL} = -1$.

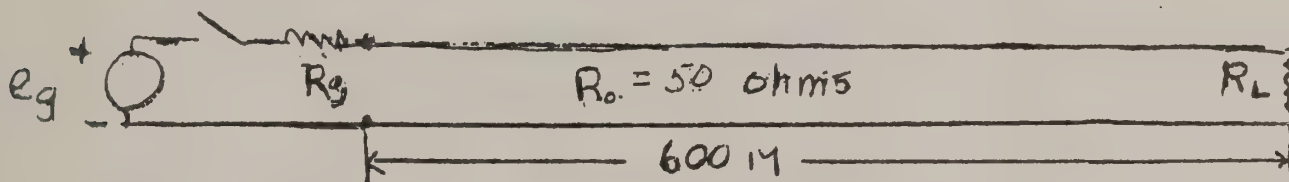


Fig. 6

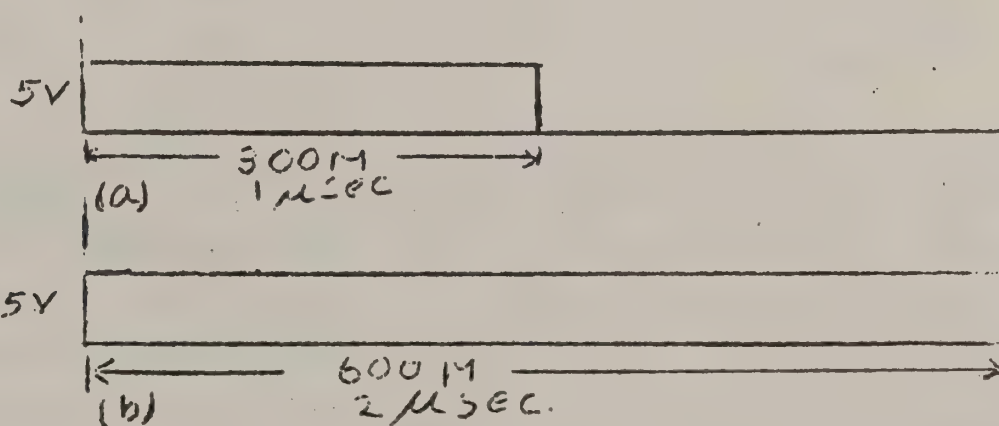


Fig. 7

The source is a 10 volt battery.

$$R_g = 50 \text{ ohms}$$

$$R_L = 50 \text{ ohms}$$

Given a loss-less line, characteristic resistance $R_0 = 50 \text{ ohms}$, 600 meters long illustrated by Fig. 6. The velocity of propagation is 300 meters per micro second or 3×10^8 meters per second.

Example 1. The source e_g is a 10 volt battery, the resistance $R_g = 50 \text{ ohms}$ and $R_L = 50 \text{ ohms}$. Figure 7 illustrates the way in which the voltage propagates down the line. When the switch s is closed the resistance to the incident wave is 50 ohms, this is true regardless of the resistance of the load. Hence the line voltage is 5 volts. At the end of 1 μsecond the voltage has propagated half way down the line. At the end of 2 $\mu\text{seconds}$ the entire line is raised to a 5 volt potential. The potential stays at 5 volts as long as the switch remains

closed. The current becomes $5/50 = .1$ ampere and propagates along with the voltage.

See page 13 for Fig. 8

Example 2. Suppose all conditions are same as in example 1 except $R_L = 150$ ohms. For this condition the voltage reflection coefficient $K_{eL} = \frac{150-50}{150+50} = +.5$ and the current reflection coefficient $K_{iL} = -.5$. Hence the incident voltage and reflected voltage at the load add up to 7.5 volts. Then at the end of 2 μ seconds +2.5 volts propagates toward the source. Since the source resistance $R_g = R_o$ there is no further reflections when the 2.5 volts reaches the source and the line remains at 7.5 volts as long as the switch is closed. The current will reach a steady value of $\frac{7.5}{150} = .05$ amperes. See Fig. 8.

See page 13 for Fig. 9

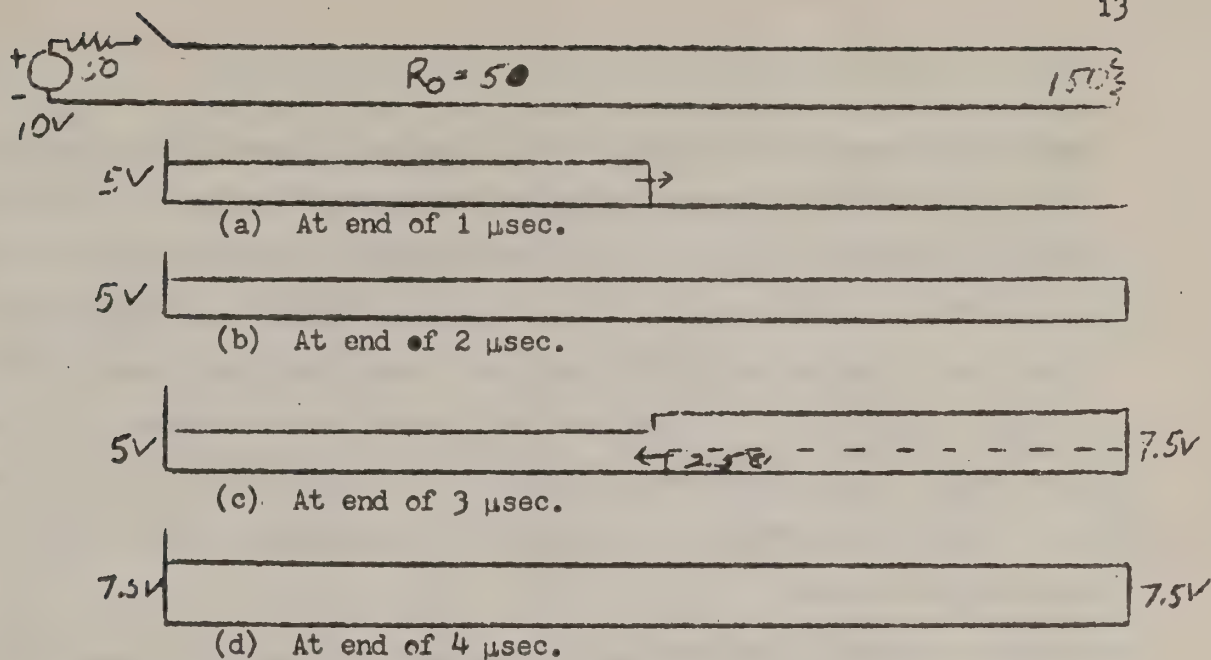


Fig. 8 $e_g = 10\text{V}$, $R_g = 50 \text{ ohms}$, $R_L = 150 \text{ ohms}$, for example 2
 $K_{eg} = 0$ and $K_{eL} = +0.5$

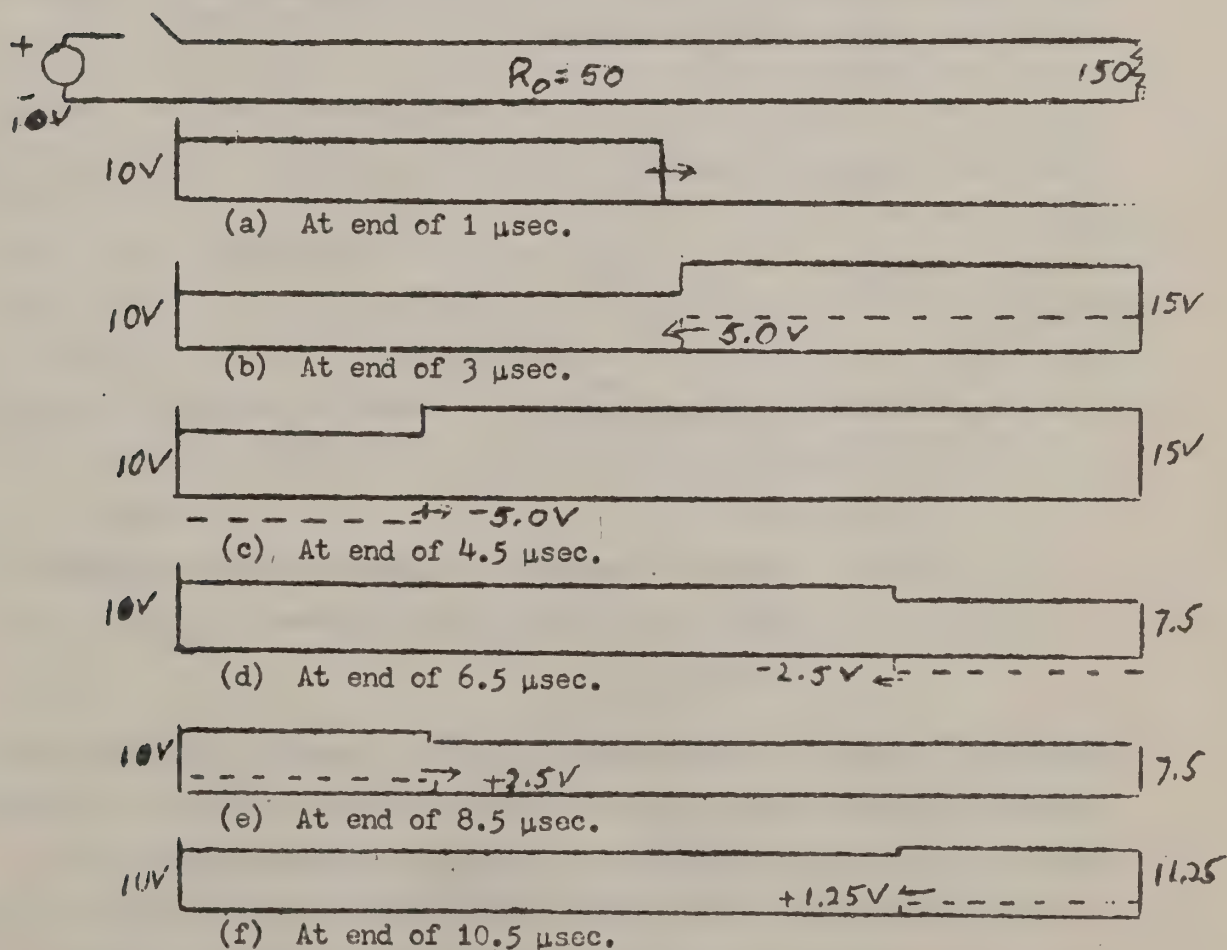


Fig. 9 $e_g = 10\text{V}$, $R_g = 0$, $R_L = 150 \text{ ohms}$, for example 3
 $K_{eg} = -1$ and $K_{eL} = +0.5$

Example 3. Suppose the conditions are the same as in example 2 except $R_g = 0$. In this case the voltage reflection coefficient at the source becomes -1.0 . The load reflection coefficient remains at $+0.5$. Now events are pictured in Fig. 9. Graph (b) shows the first reflection of $+ .5 \times 10 = 5$ volts travelling toward the source. At the source -5 volts are reflected and travel toward the load. Graph (d) shows the second reflection of -2.5 volts travelling toward the source where $+2.5$ volts are reflected. The third reflection results in $+1.25$ volts travelling toward the load. Thus the reflections are dying out and the steady state voltage of the entire line becomes 10 volts.

Example 4. Suppose the conditions are the same as in example 1 except the 10 volt battery (step voltage) is replaced by a 20 volt pulse of $1 \mu\text{sec}$. duration. Since $R_L = R_0$ this pulse which will reach the load in $2 \mu\text{seconds}$, will be dissipated entirely in the load resistor, no reflection.

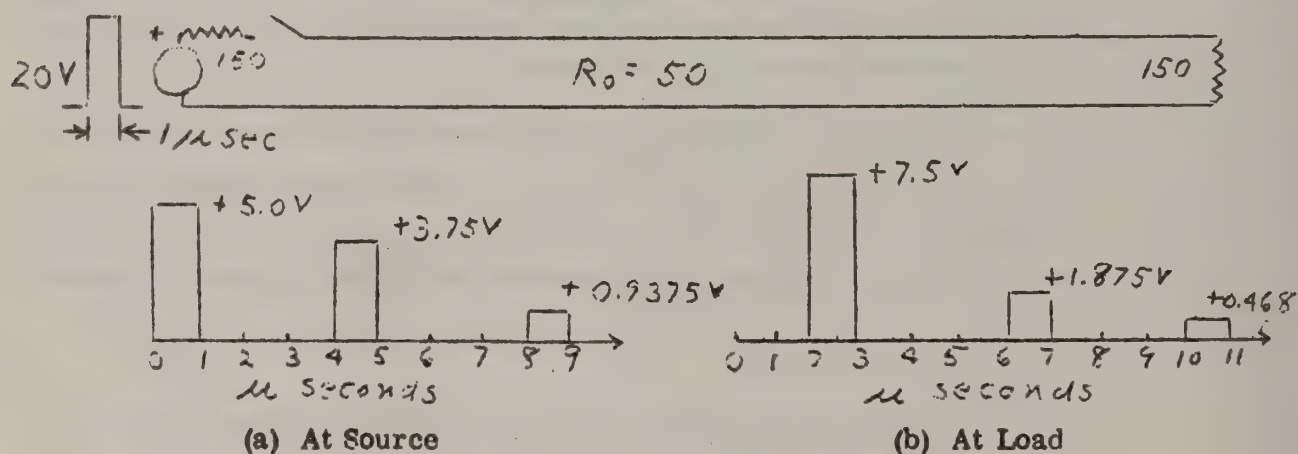


Fig. 10 For each reflection the incident and reflected voltages add to give the total voltage as shown. $K_{eg} = K_{eL} = +.5$ for example 5

Example 5. Suppose the conditions are the same as example 4 except the load resistance is 150 ohms and the source resistance is also 150 ohms. The voltage reflection coefficients at the load and at the source are both equal to 0.5. A $1 \mu\text{sec}$ pulse of 5 volts $[= 50 \times 20 / (150 + 50)]$ travels toward the load and reaches the load in $2 \mu\text{seconds}$. It is reflected at the load as a $+2.5$ volt pulse which travels toward and reaches the source end in another $2 \mu\text{seconds}$. It is reflected at the source as 1.25 volts travels toward the load and reaches the load in an

additional 2 μsec and so on. Figure 10 represents the total voltage source and load for the first several microseconds.

Example 6 Lumped element transmission lines are often used in radar equipment to provide high voltage pulses of short duration for modulation. Figure 11 shows a lumped constant line that is initially charged to 5 KV. Let us determine the load voltage when the switch is closed.

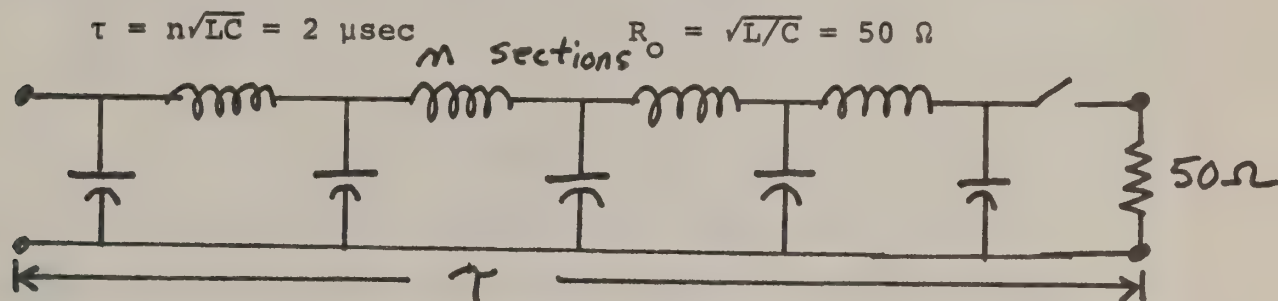


Fig. 11

To determine the initial load voltage at $t = 0+$ we consider the line to exhibit at the load terminals a Thevenin source of 5 KV in series with a 50Ω resistance. Hence the initial load voltage is 2.5 KV. The line voltage has thus dropped by 2.5 KV at the load end. This disturbance then propagates down the line toward the open end discharging line capacitance by 2.5 KV as it moves. Upon reflection from the open end ($K=+1$) the -2.5 KV step proceeds back toward the load dropping the line voltage to zero as it goes. Since the line is $\tau = 2 \mu\text{sec}$ long the resultant load voltage is shown in Fig. 12.

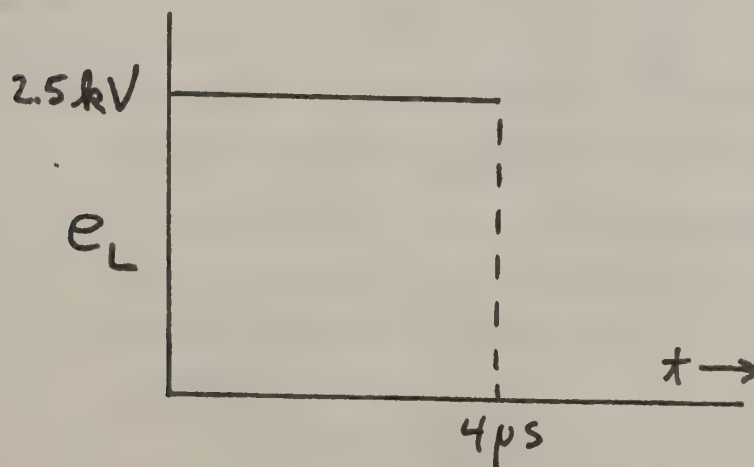


Fig. 12

Example 7 Suppose the system shown in Fig. 13 has reached a steady state condition when at $t=0$ the switch is opened. Let us determine the load voltage e_L as a function of time.

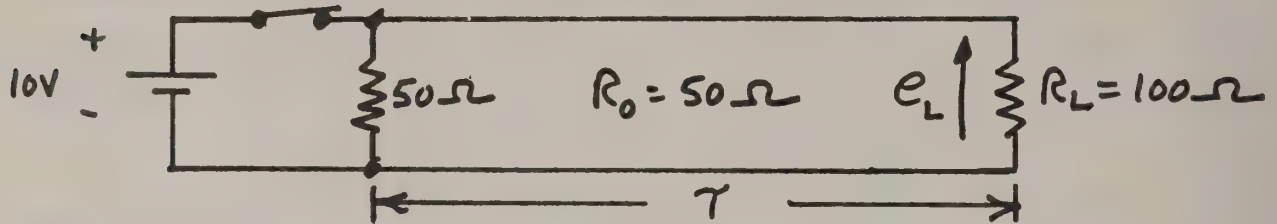


Fig. 13

at $t=0^-$, $i_{IN} = \frac{10}{50} + \frac{10}{100} = \frac{30}{100}$ a. Hence at $t=0^+$ we consider the circuit of Fig. 14 wherein a step current of $-\frac{30}{100}$ a is

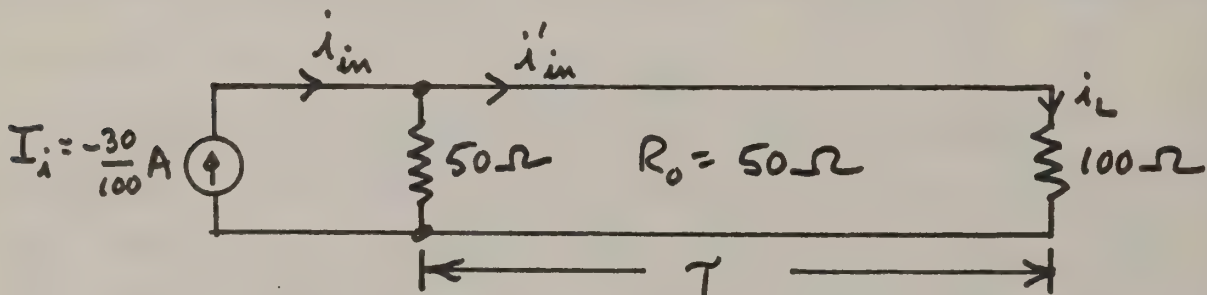


Fig. 14

applied. Since $R_0 = 50\Omega$ this current divides equally at the input resulting in $i'_{IN} = -\frac{30}{200}$ a. After τ seconds this current reaches the 100Ω load where it sees $K_{IL} = -\frac{1}{3}$. A reflected current step of $+\frac{10}{200}$ a then proceeds toward the generator end where it is absorbed τ seconds later. The resultant load current and voltage are shown in Fig. 15 and 16.

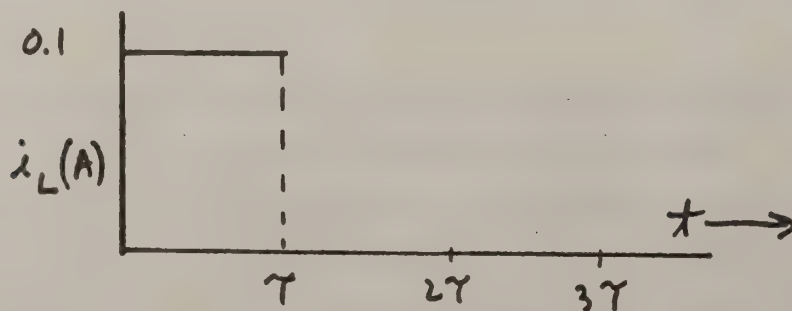


Fig. 15

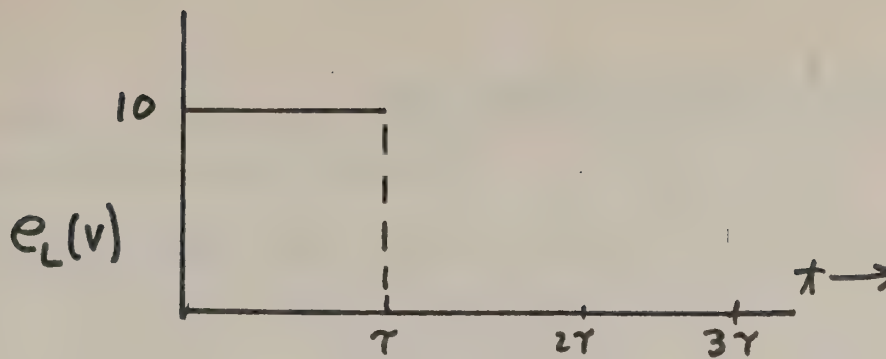


Fig. 16

5. Lines with other than resistive loads

Under transient excitation lines terminated in inductance or capacitance will have reflection coefficients that vary with time. (Note that impedance is a steady state sinusoidal concept and has no application here.) For example, to a voltage step, an inductor looks initially like an open circuit while as $t \rightarrow \infty$ it becomes a short. Hence the voltages and currents on non-resistively terminated lines will vary not only due to time delays introduced by the line but also due to the finite time required to establish a current in an inductor or a voltage across a capacitor.

Consider the circuit of Fig. 17. Let us determine the load reflection coefficient, load voltage and input voltage for this circuit after the switch is closed.

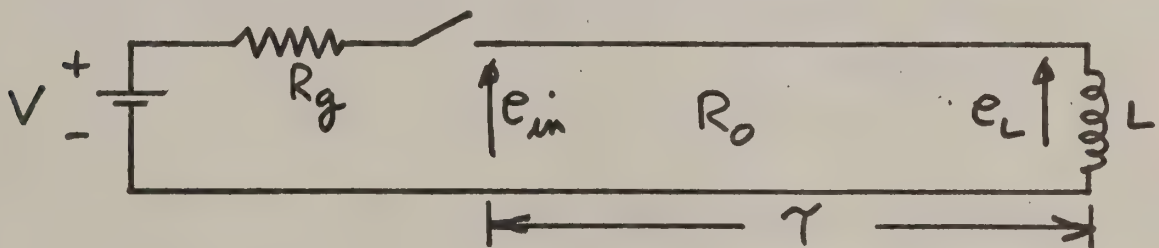


Fig. 17

To find the reflection coefficient we proceed as in the resistive case by expressing the load voltage and current as constrained by the line equations. At $z = 0$,

$$e_L = e_1 + e_2 \quad 5$$

$$i_L = e_1/R_0 - e_2/R_0 \quad 6$$

Also the load itself dictates that

$$e_L = L \frac{di_L}{dt} \quad 7$$

Combining Eqs. 5, 6 and 7 results in the differential equation

$$e_1 + e_2 = \frac{L}{R_0} \left(\frac{\partial e_1}{\partial t} - \frac{\partial e_2}{\partial t} \right) \quad 8$$

However, at $t = 0^+$

$$\frac{\partial e_1}{\partial t} = 0 \quad 9$$

which simplifies Eq. 8, resulting in

$$\frac{L}{R_0} \frac{\partial e_2}{\partial t} + e_2 + e_1 = 0 \quad 10$$

with e_1 constant in time. This equation has the solution

$$e_1 + e_2 = e_0 e^{-R_0 t/L} \quad 11$$

It is convenient to measure time from the arrival of the incident wave at the load, i.e., the switch is closed at $t = -\tau$ seconds. Hence, at $t = 0^+$ the load appears as an open circuit and at $z = 0$

$$e_2 = e_1 \quad 12$$

Equation 11 then yields

$$e_0 = 2e_1 \quad 13$$

Substituting Eq. 13 into Eq. 11 allows the determination of the reflection coefficient at the load as a function of time, viz,

$$\rho_L = \frac{e_2}{e_1} = 2e^{-tR_0/L} - 1 \quad 14$$

Figure 18 shows Eq. 14 as well as the load and input voltages as a function of time.

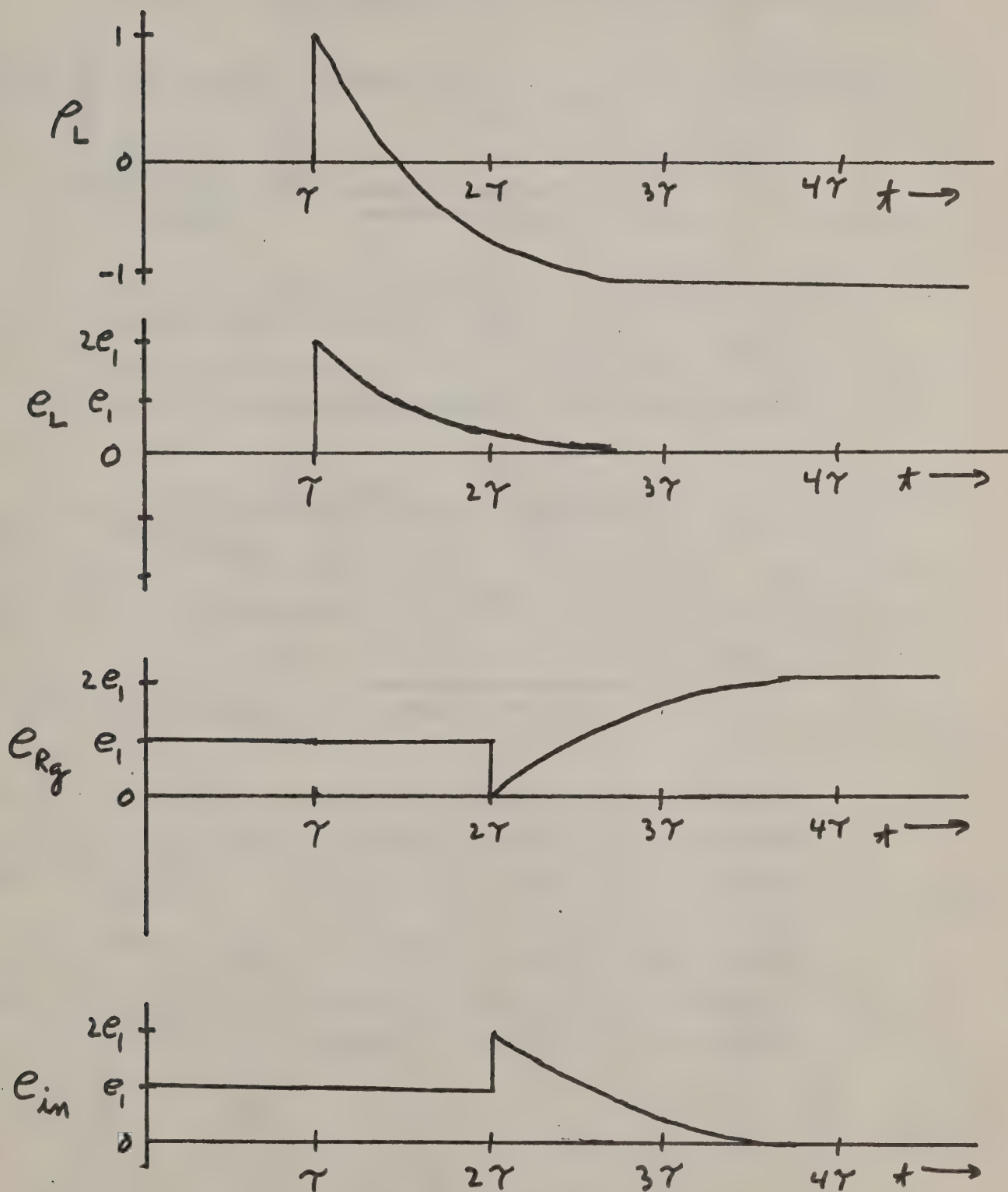
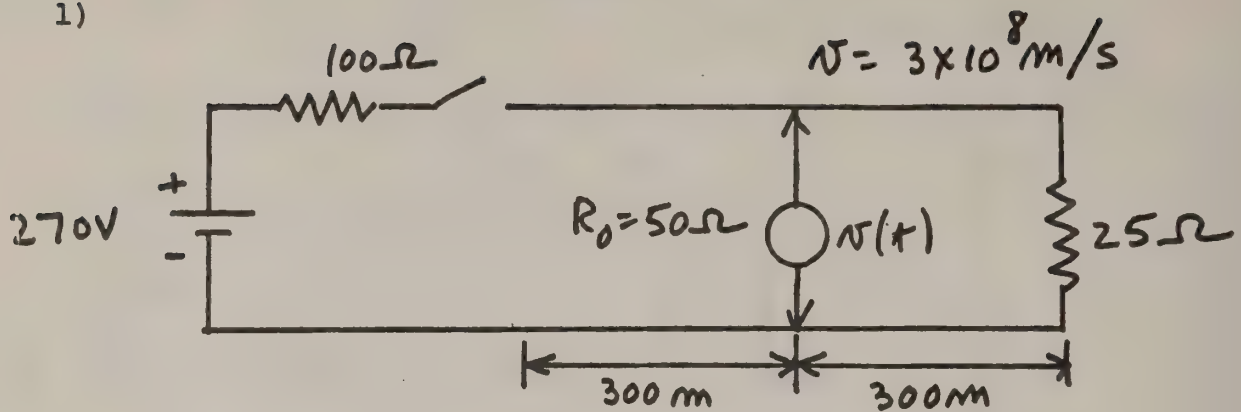


Fig. 18

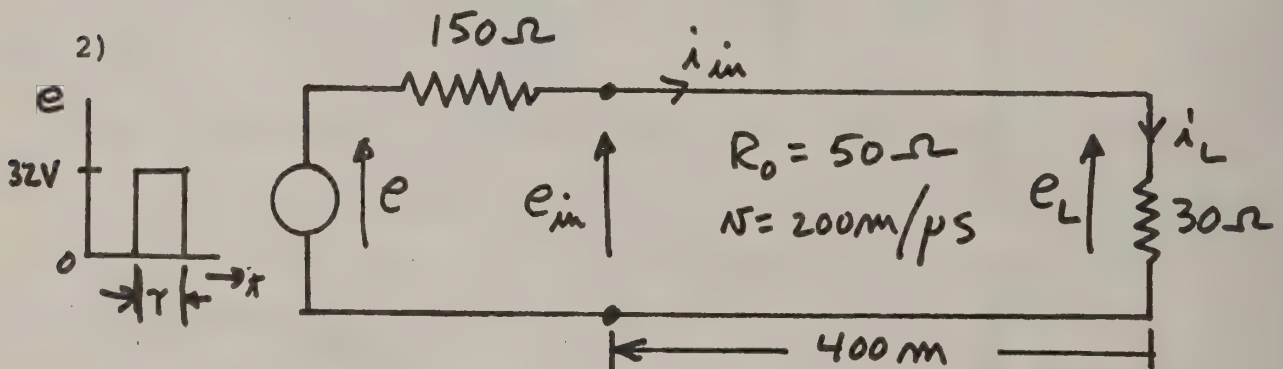
Problems

1)



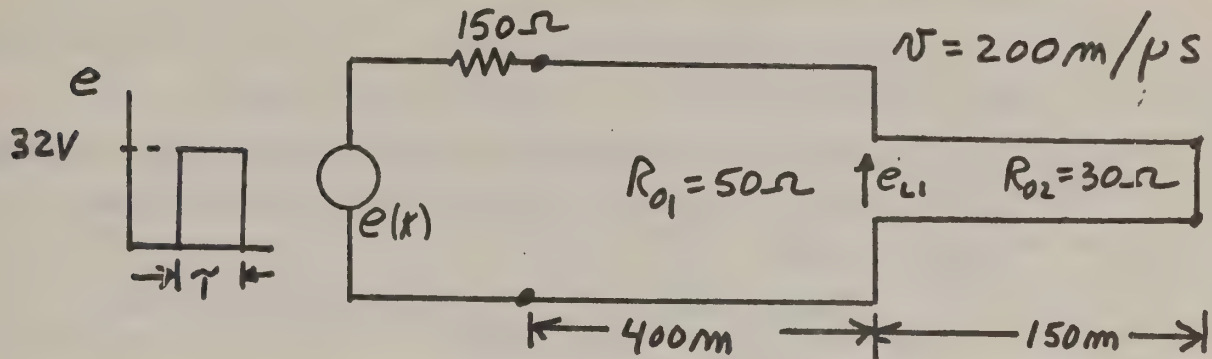
- Determine and plot the voltmeter reading $v(t)$ for the period $t = 0$ to $10 \mu\text{sec}$.
- What is the final voltage across R_L ?

2)

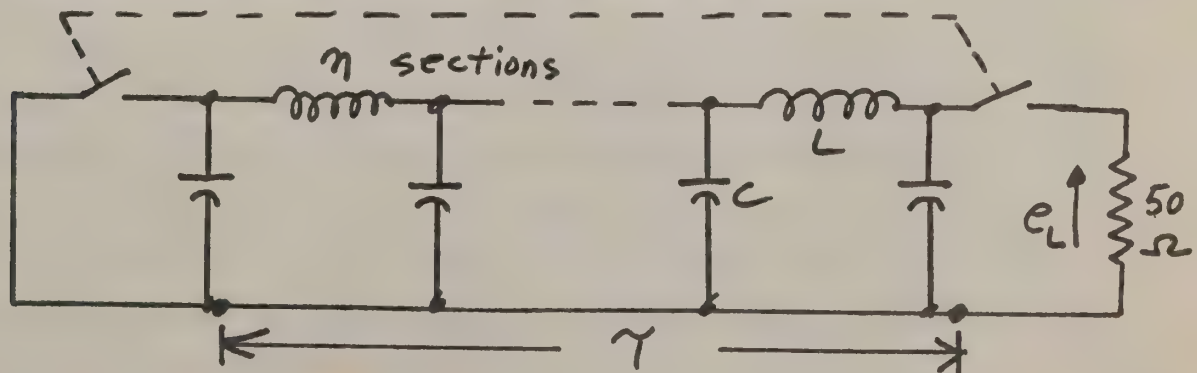


- Determine and plot the input voltage $e_{IN}(t)$ and the input current $i_{IN}(t)$ for the period $t = 0$ to $10 \mu\text{sec}$ if $\tau = 1 \mu\text{sec}$.
- Repeat (a) for the load voltage $e_L(t)$ and the load current $i_L(t)$.
- Repeat parts (a) and (b) for $\tau = 6 \mu\text{sec}$.
- Determine and plot the voltage and current at the midpoint of the line for $0 \leq t \leq 10 \mu\text{sec}$ for $\tau = 1 \mu\text{sec}$.

- 3) Determine and plot $e_{L_1}(t)$ for the period $0 \leq t \leq 10$ sec for $\tau = 1 \mu\text{sec}$.



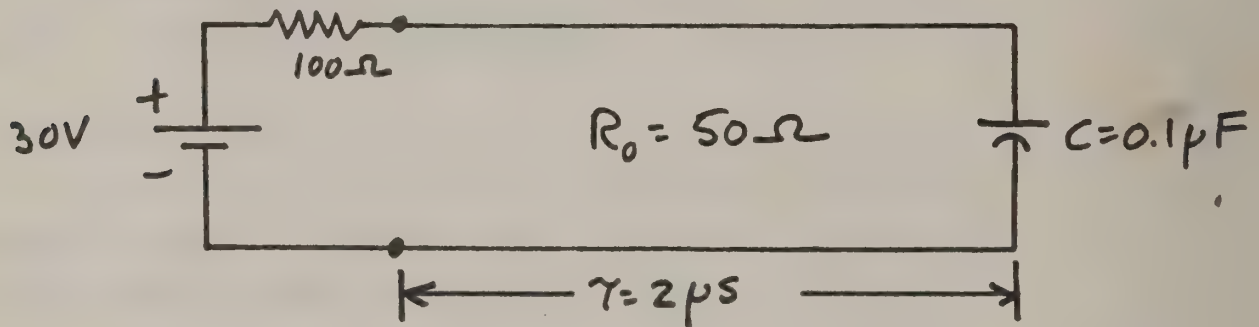
- 4) Find the voltage as a function of time across the 50Ω resistor in Example 7, Fig. 13.
- 5) Assume in Example 7, Fig. 13 that the switch is open and the line initially uncharged. Find and plot $v_L(t)$ for $t \geq 0$.
- 6) A common dielectric for insulating coaxial lines is polyethylene with $\epsilon_r = 2.25$. Show that $v = 200 \text{ m}/\mu\text{sec}$.
- 7) Explain how an oscilloscope, a pulse generator and a precision variable resistor can be used to determine the characteristic resistance of a lossless line.
- 8) A typical coaxial line is designated as RG-8A/U. For this line $R_0 = 50\Omega$ and $v = 200 \text{ m}/\mu\text{sec}$. Find the per meter values of L and C .
- 9) Show in Example 6, Fig. 11 that indeed the delay time $\tau = n\sqrt{LC}$ where L and C are the values of the lumped constants. Determine the values of L and C for this example.
- 10) Suppose Fig. 11, Example 6 is modified as shown.



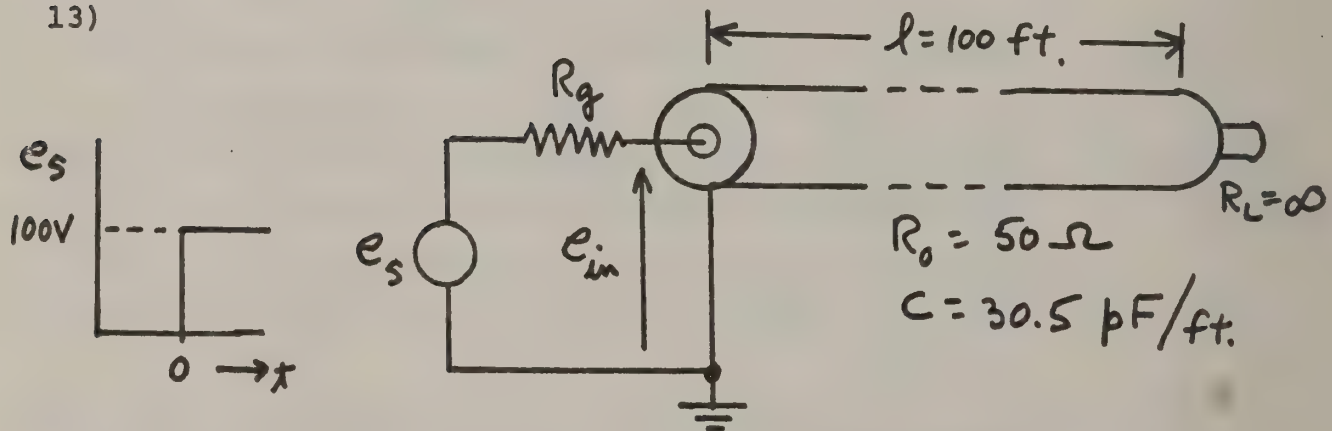
The line is again initially charged to 5 KV. Switches S_1 and S_2 are closed simultaneously at $t=0$. Find $e_L(t)$.

11) Determine e_{in} in Section 5, Fig. 17 if $R_0 = 50\Omega$, $R_g = 100\Omega$ and $V = 30$ volts.

(12) Find and plot the reflection coefficient, load voltage and input voltage as a function of time for the line shown below.



13)



Determine and plot $e_{IN}(t)$ at the input of an open circuit 100 foot length of RG-8A/U coax if $R_g = 1000\Omega$. Find the value of an equivalent capacitor with which the line could be replaced to give essentially the same waveform. Compare the value of this capacitor with the total line capacitance. Replot $e_{IN}(t)$ when $R_g = 50\Omega$. (Hint: $\sum_{n=0}^{\infty} \rho^n = \frac{1-\rho^n}{1-\rho}$) Ans: $C = 2940\text{ pf}$

Transmission Line Theory. (General)

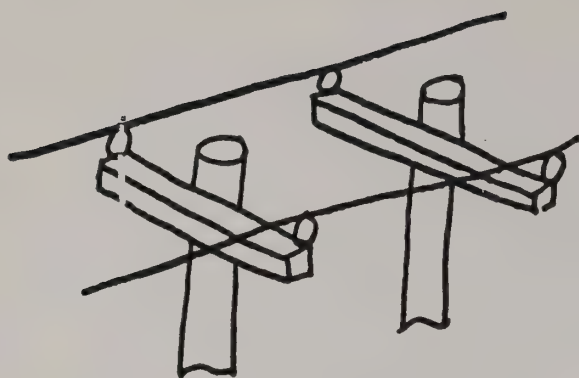
1. Introductory

The transmission lines treated in these notes consists of two conductors, usually copper or aluminum, arranged in various configurations some of which are illustrated symbolically by Fig.1. The purpose of a transmission line is to guide electrical energy in some form, such as 60 cycle energy for home and commercial purposes or modulated energy in the form of telephone conversation and others, from one location to another. The conductors of the transmission line do not transmit the energy but serve only as guides for the energy which is transmitted in the electromagnetic field surrounding the conductors. This is the only concept that satisfies the Maxwell field equations and agrees with other systems of energy transmission such as hollow wave guides where only one conductor exists and radio transmission where there are no conductors.

The mathematical treatment and physical behavior of a transmission line depends upon whether the line is electrically short or long. Actual physical length and electrical length are two different quantities when deciding whether a line is short or long. A transmission line is 100 meters long physically. At 60 cycles per second it is very short electrically. At 10^8 cycles per second it is very long electrically. Electrical length is expressed in terms of wavelength for a particular operating frequency. Wave length is the distance for which a voltage, or current, undergoes a 360° phase shift from one end of the line to the other. One wave length at 60 cps is 3100 miles. Whereas it is only 3 meters at 10^8 cps.

A transmission line is a linear circuit consisting of resistance R, inductance L, capacitance C and leakage conductance G uniformly distributed throughout the length of the line. The resistance and inductance are in series with the line conductors and the capacitance and leakage conductance are in shunt as illustrated in Fig. 2. Because of the distributed nature of these parameters the current and voltage on a line undergo continual changes in magnitude and phase along the line in relation to the current and voltage at some reference point such as the sending end or the receiving end.

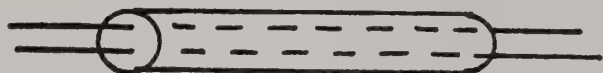
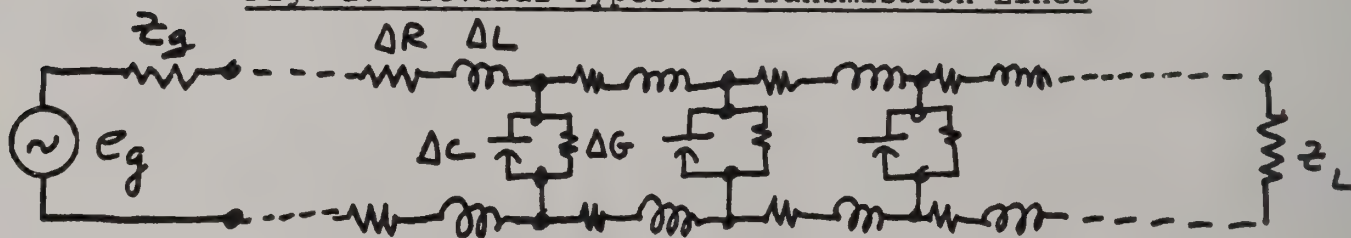
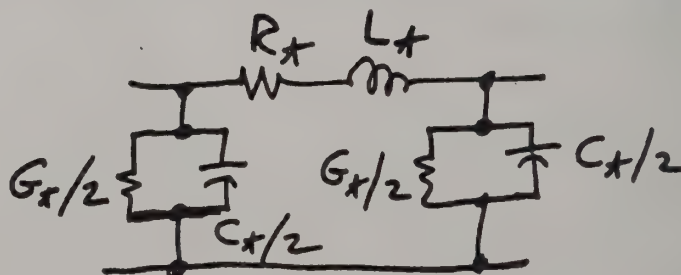
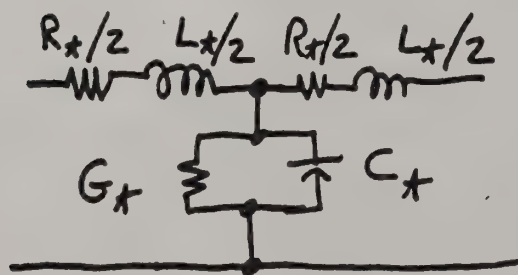
A very short, electrically, length of transmission line may be represented as a T or π network as shown in Fig. 3. For example 50 miles of 60 cycle power line is only .0155 wave lengths long and may therefore be represented as a single T or π network. For a power line only a few miles long it is generally sufficient



a. Open wire line



b. Coaxial line

c. Shielded pair line.
Outside shield usually
copper and grounded.d. Cable type line. Sheath may be
metallic. Some cables contain
several pairs of conductors.Fig. 1. Several Types of Transmission LinesFig. 2. A very short piece of transmission line between the
source e_g and the load z_L , showing how R , L , C and G are
disposed to form a circuit of uniformly distributed parameters.Fig. 3. A very short transmission line represented by a T or π
network. The parameters R_t , L_t , C_t and G_t are for the entire
length of line.

to replace the line by its overall resistance and inductance and neglect its capacitance and leakage conductance.

When a line becomes of the order of .05 wave lengths or longer and there is only one frequency of interest the line may be represented by a single T or π network whose parameters are determined from the open and short circuit impedances of the line. However, open and short circuit impedances must be determined either from measurements or from mathematical theory which takes into consideration the uniform distribution of the line parameters. Thus for a line that cannot be represented by a T or π network without the use of the open and short circuit impedance it becomes simpler to treat the line as a uniformly distributed parameter circuit and derive and solve the differential equation which apply.

2. Derivation of the Differential Equations for the Transmission Line.

In order to arrive at a pair of equations which express the relations for voltage and current at any point on a transmission line it is necessary to first derive the fundamental differential equations for the line. These equations are then solved for voltage and current in terms of the line parameters, the distance to a point in question and the terminal conditions.

The differential equations are developed by applying the e.m.f. and current laws to an infinitesimal portion of the line. Fig. 4 represents an infinitesimal portion of a transmission line. Since the current in one conductor is equal to, but opposite in direction, at any instant of time to the current in the other conductor, all line resistance and inductance of the infinitesimal portion of line is placed in the upper branch as shown.

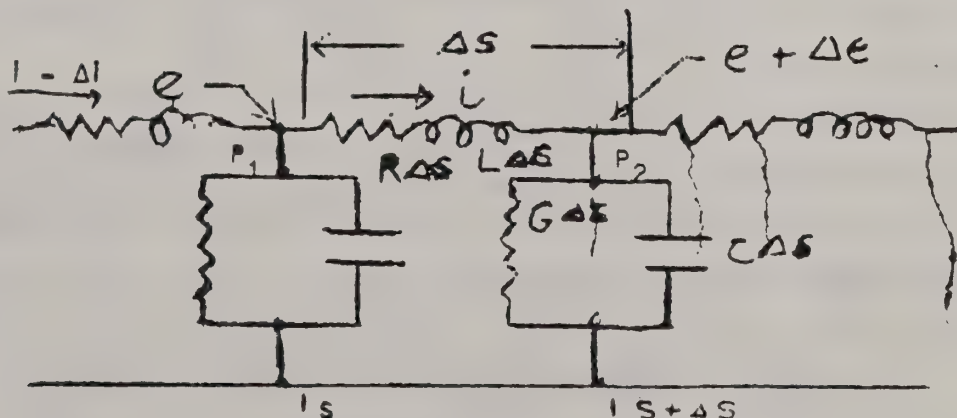


Fig. 4 An infinitesimal length of transmission line.
The parameters R , L , C and G are for a unit length of line.

Referring to Fig. 4 the difference in potential between the line conductors at point P_2 is equal to the potential difference e at P_1 plus Δe where Δe is the incremental change in potential difference in going from P_1 to P_2 because of the current i in $R\Delta s$ and $L\Delta s$. In equation form this becomes

$$e - (Ri + L \frac{\partial i}{\partial t}) \Delta s = e + \Delta e$$

Which gives, as $\Delta s \rightarrow 0$

$$\frac{\partial e}{\partial s} = - (Ri + L \frac{\partial i}{\partial t}) \quad (1)$$

In the infinitesimal portion of line from P_1 to P_2 the current i and potential e are changing with respect to both time and distance, thus the partial derivatives.

In a similar manner as the above the current to the right of P_2 differs from the current between P_1 and P_2 by Δi where Δi is the incremental change in current due to the shunt paths $G\Delta s$ and $C\Delta s$. In equation form

$$i = \Delta i - \Delta i - (Ge + C \frac{\partial e}{\partial t}) \Delta s$$

Which gives, as $\Delta s \rightarrow 0$

$$\frac{\partial i}{\partial s} = - (Ge + C \frac{\partial e}{\partial t}) \quad (2)$$

Equations 1 and 2 are the fundamental differential equations for a transmission line. The equations give the space rate-of-change of e and i respectively, and at each point on the line e and i are changing with respect to time. This indicates that e and i are propagated along the line in a wave pattern. This will become clearer when the differential equations are solved and physical interpretations are given for the resulting solutions.

Solutions of equations 1 and 2 are best carried out by first obtaining two equations, each of which is expressed in terms of a single dependent variable, i.e. one equation containing only e and the other containing only i . This is carried out as follows.

Differentiation of 1 with respect to s gives

$$\frac{\partial^2 e}{\partial s^2} = - (R \frac{\partial i}{\partial s} + L \frac{\partial^2 i}{\partial t \partial s}) \quad (3)$$

Differentiation of 2 with respect to t gives

$$\frac{\partial^2 i}{\partial t \partial s} = (G \frac{\partial e}{\partial t} + C \frac{\partial^2 e}{\partial t^2}) \quad (4)$$

Now substitute $\frac{\partial i}{\partial s}$ from equation 2 and $\frac{\partial^2 i}{\partial t \partial s}$ from equation 4 into equations 3 and get

$$\frac{\partial^2 e}{\partial s^2} = RGe + (RC + LG) \frac{\partial e}{\partial t} + LC \frac{\partial^2 e}{\partial t^2} \quad (5)$$

In like manner

$$\frac{\partial^2 i}{\partial s^2} = RG i + (RC + LG) \frac{\partial i}{\partial t} + LC \frac{\partial^2 i}{\partial t^2} \quad (6)$$

These are the second order differential equations that govern the behavior of e and i for the transmission line.

Except for the special case which is the steady state solution when e and i are sinusoidal functions of time the solutions of equations 5 and 6 are quite complicated both mathematically and physically. Hence in order to acquire some physical insight into the nature of e and i before proceeding with steady state sinusoidal solutions equation 5 and 6 will be modified to represent the loss-less line condition, $i, e, R = G = 0$. In this sense they become

$$\frac{\partial^2 e}{\partial s^2} = LC \frac{\partial^2 e}{\partial t^2} \quad (7)$$

$$\frac{\partial^2 i}{\partial s^2} = LC \frac{\partial^2 i}{\partial t^2} \quad (8)$$

3. Travelling Waves.

Equations 7 and 8 are generally known as the wave equations for loss-less transmission lines. Just why these equations portray wave motion may be seen by examining in some detail their steady state solutions when e and i are sinusoidal functions of time. An assumed solution for equation 7 is

$$e = e_1 \cos (wt - \beta s) + e_2 \cos (wt + \beta s) \quad (9)$$

That this is a solution may be shown as follows:

$$\frac{\partial e}{\partial s} = + \beta e_1 \sin (\omega t - \beta s) - \beta e_2 \sin (\omega t + \beta s)$$

$$\begin{aligned} \frac{\partial^2 e}{\partial s^2} &= - \beta^2 e_1 \cos (\omega t - \beta s) - \beta^2 e_2 \cos (\omega t + \beta s) \\ &= \beta^2 [-e_1 \cos (\omega t - \beta s) - e_2 \cos (\omega t + \beta s)] \end{aligned}$$

Likewise

$$\begin{aligned} \frac{\partial^2 e}{\partial t^2} &= - \omega^2 e_1 \cos (\omega t - \beta s) - \omega^2 e_2 \cos (\omega t + \beta s) \\ &= \omega^2 [-e_1 \cos (\omega t - \beta s) - e_2 \cos (\omega t + \beta s)] \end{aligned}$$

Then if $\beta = \omega \sqrt{LC}$ equation 9 becomes a solution of equation 7. Now by graphing the first term of equation 9 in Fig. 5 for several instants of time it is seen that this term represents a wave moving in the positive direction of s . In a similar manner the second term of equation 9 represents a wave moving in the negative s direction. For one complete period, i.e. $1/\text{frequency}$, the wave moves a distance called one wave length λ . Thus

$$s_\lambda = \lambda = \frac{2\pi}{\beta} \quad \text{OR} \quad \beta = \frac{2\pi}{\lambda} \quad (10)$$

Now s = velocity of wave multiplied by time or

$$s_\lambda = \lambda = \text{vel.} \cdot t = \text{vel.} / f = \frac{2\pi}{\beta}$$

$$\text{Hence} \quad \text{vel} = \frac{2\pi f}{\beta} \quad (11)$$

$$\text{and} \quad \beta = 2\pi f \sqrt{LC} \quad (12)$$

Consequently velocity = $\frac{1}{\sqrt{LC}} = v_p$ and is called the phase velocity of the wave. For a transmission line with air or vacuum dielectric it may be shown that if the self inductance due to magnetic flux linkages inside the conductors is neglected $1/\sqrt{LC} = c$ the velocity of light in free space. For any lossless line $v_p = c/\sqrt{\epsilon_r}$ where ϵ_r is the dielectric constant of the medium around the line conductors.

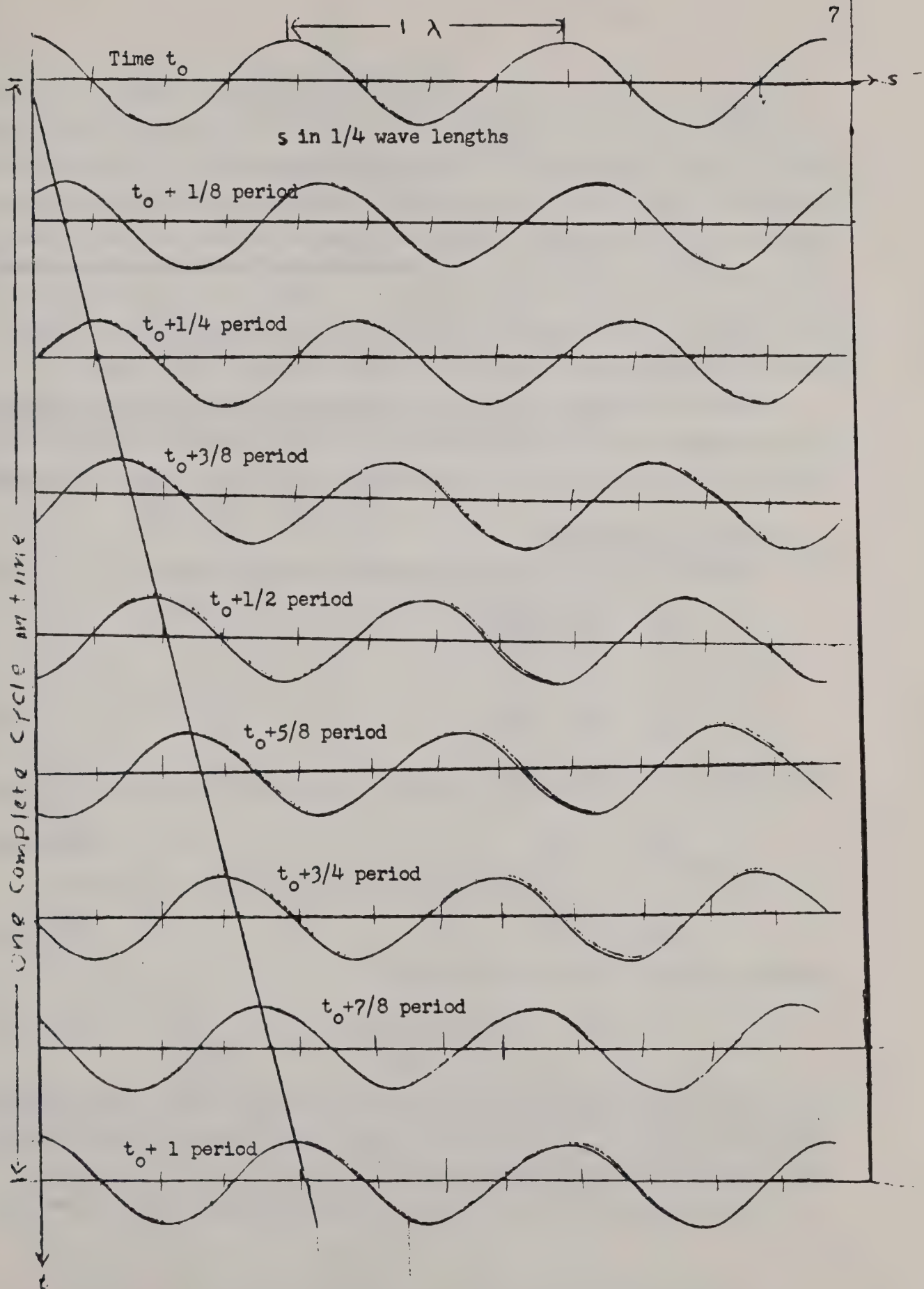


Fig. 5. Wave moves to right 1λ in 1 cycle of time

Assume now a line so terminated that only the first term of equation 9 exists. This is possible as will be seen later. Suppose two oscilloscopes were connected across the line s meters apart. Both oscilloscopes would show sinusoidal voltage - time waves. However, the voltage-time wave shown on the oscilloscope farthest from the source end would be βs degrees lagging the voltage-time wave shown by the other oscilloscope. Suppose the oscilloscope nearest the source were connected to the line at the instant the voltage was e_a volts and then moved along the line at a velocity v_p . The oscilloscope would continue to show e_a volts. These observations help to show that a travelling wave exists on the line.

When the resistance and leakage conductance are not zero, as is the case for a realizable line, the travelling waves suffer attenuation along the line. This will be shown later when the equations for the general case are discussed.

4. Transient considerations.

Returning now to the basic differential equations namely

$$\frac{\partial e}{\partial s} = -(Ri + L \frac{\partial i}{\partial t})$$

$$\frac{\partial i}{\partial s} = -(Ge + C \frac{\partial e}{\partial t})$$

and setting $R = G = 0$ for the loss less case there results

$$\frac{\partial e}{\partial s} = -L \frac{\partial i}{\partial t} \quad (13)$$

$$\frac{\partial i}{\partial s} = -C \frac{\partial e}{\partial t} \quad (14)$$

A general solution of equation 13 for e is

$$e = e_1 f(t - \frac{s}{v}) + e_2 f(t + \frac{s}{v}) \quad (15)$$

where $v = 1/\sqrt{LC}$ and is the velocity of propagation as was found for the steady state solution when e and i vary sinusoidally with time. It may be shown from fundamental dimension that $1/\sqrt{LC}$ is velocity.

Now examining, in detail the first term of equation 15 it is seen that

$$\begin{aligned}\frac{\partial e}{\partial s} &= e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \frac{\partial(t - \frac{s}{v})}{\partial s} \\ &= e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \left(-\frac{1}{v}\right)\end{aligned}\quad (16)$$

Assume now the corresponding solution for i . i.e.

$$i = \frac{e_1}{R_o} f(t - \frac{s}{v}) - \frac{e_2}{R_o} f(t + \frac{s}{v}) \quad (17)$$

and taking the partial derivative of the first term, i.e.

$$\frac{\partial i}{\partial t} = \frac{e_1}{R_o} \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})}$$

Then

$$e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \left(-\frac{1}{v}\right) = -L \frac{e_1}{R_o} \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \quad (18)$$

$$\text{This is true if } R_o = \sqrt{\frac{L}{C}} \quad (19)$$

The $\sqrt{\frac{L}{C}}$ is called the characteristic resistance of the loss-less line and is designated as R_o or R_c .

Proceeding in a similar manner for ^{the} second terms of e and i ^{again} results in

$$R_o = \sqrt{\frac{L}{C}}.$$

$$\text{Then } i = \frac{e_1}{\sqrt{L/C}} f(t - \frac{s}{v}) - \frac{e_2}{\sqrt{L/C}} f(t + \frac{s}{v}) \quad (20)$$

Now inasmuch as $e_1 f(t - \frac{s}{v})$ is a wave travelling in the positive direction of s and $e_2 f(t + \frac{s}{v})$ is travelling in the negative s direction it is proper to call $e_1 f(t - \frac{s}{v})$ an incident wave and $e_2 f(t + \frac{s}{v})$ a reflected wave.

Hence

$$e = e_1 + e_r \quad \text{where} \quad (21)$$

$$e_1 = e_1 f(t - \frac{s}{v}) \quad \text{and} \quad e_r = e_2 f(t + \frac{s}{v})$$

Likewise

$$i = i_1 + i_r \quad \text{where} \quad (22)$$

$$i_1 = \frac{e_1}{R_o} f(t - \frac{s}{v}) \quad \text{and} \quad i_r = -\frac{e_2}{R_o} f(t + \frac{s}{v})$$

$$\text{where } R_o = \sqrt{L/C}$$

Suppose now a loss-less line is terminated in a resistance R_L , then $e_L = i_L R_L$. Hence

$$\begin{aligned} e_L &= e_{1L} + e_{rL} \\ i_L &= \frac{e_{1L}}{R_o} - \frac{e_{rL}}{R_o} \end{aligned} \quad (23)$$

The solution of these two equations yields

$$\frac{e_{rL}}{e_{1L}} = \frac{R_L - R_o}{R_L + R_o} = K_{eL} \quad (24)$$

$$\frac{i_{rL}}{i_{1L}} = -\frac{R_L - R_o}{R_L + R_o} = K_{iL} = -K_{eL}$$

Equations 24 give the voltage reflection coefficient K_e and current reflection coefficient K_i both at ^{the} load resistance R_L . In other words

$$e_{rL} = K_{eL} e_{1L} \quad \text{and} \quad i_{rL} = K_{iL} i_{1L}$$

For the case in which $R_L = R_0$, $K_{eL} = K_{iL} = 0$ and there is no reflection. That is, all of the energy that reaches the load is dissipated in the load.

When the line is shorted at the load $R_L = 0$, $K_{eL} = -1$ and $K_{iL} = +1$. When the line is open at the load $R_L = \infty$, $K_{eL} = +1$ and $K_{iL} = -1$.

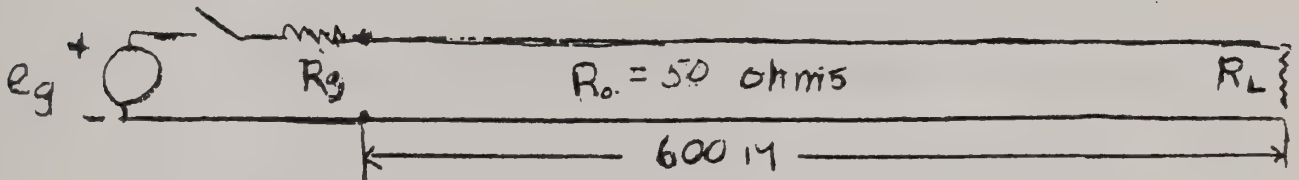


Fig. 6

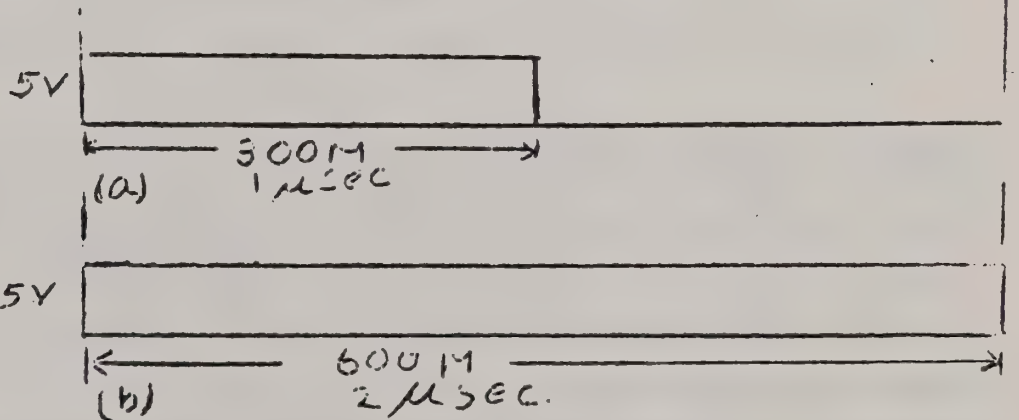


Fig. 7

The source is a 10 volt battery.

$$R_g = 50 \text{ ohms}$$

$$R_L = 50 \text{ ohms}$$

Given a loss-less line, characteristic resistance $R_0 = 50$ ohms, 600 meters long illustrated by Fig. 6. The velocity of propagation is 300 meters per micro second or 3×10^8 meters per second.

Example 1. The source e_g is a 10 volt battery, the resistance $R_g = 50$ ohms and $R_L = 50$ ohms. Figure 7 illustrates the way in which the voltage propagates down the line. When the switch s is closed the resistance to the incident wave is 50 ohms, this is true regardless of the resistance of the load. Hence the line voltage is 5 volts. At the end of 1 μ second the voltage has propagated half way down the line. At the end of 2 μ seconds the entire line is raised to a 5 volt potential. The potential stays at 5 volts as long as the switch remains

closed. The current becomes $5/50 = .1$ ampere and propagates along with the voltage.

See page 13 for Fig. 8

Example 2. Suppose all conditions are same as in example 1 except $R_L = 150$ ohms. For this condition the voltage reflection coefficient $K_{eL} = \frac{150-50}{150+50} = +.5$ and the current reflection coefficient $K_{iL} = -.5$. Hence the incident voltage and reflected voltage at the load add up to 7.5 volts. Then at the end of 2 μ seconds +2.5 volts propagates toward the source. Since the source resistance $R_g = R_o$ there is no further reflections when the 2.5 volts reaches the source and the line remains at 7.5 volts as long as the switch is closed. The current will reach a steady value of $\frac{7.5}{150} = .05$ amperes. See Fig. 8.

See page 13 for Fig. 9

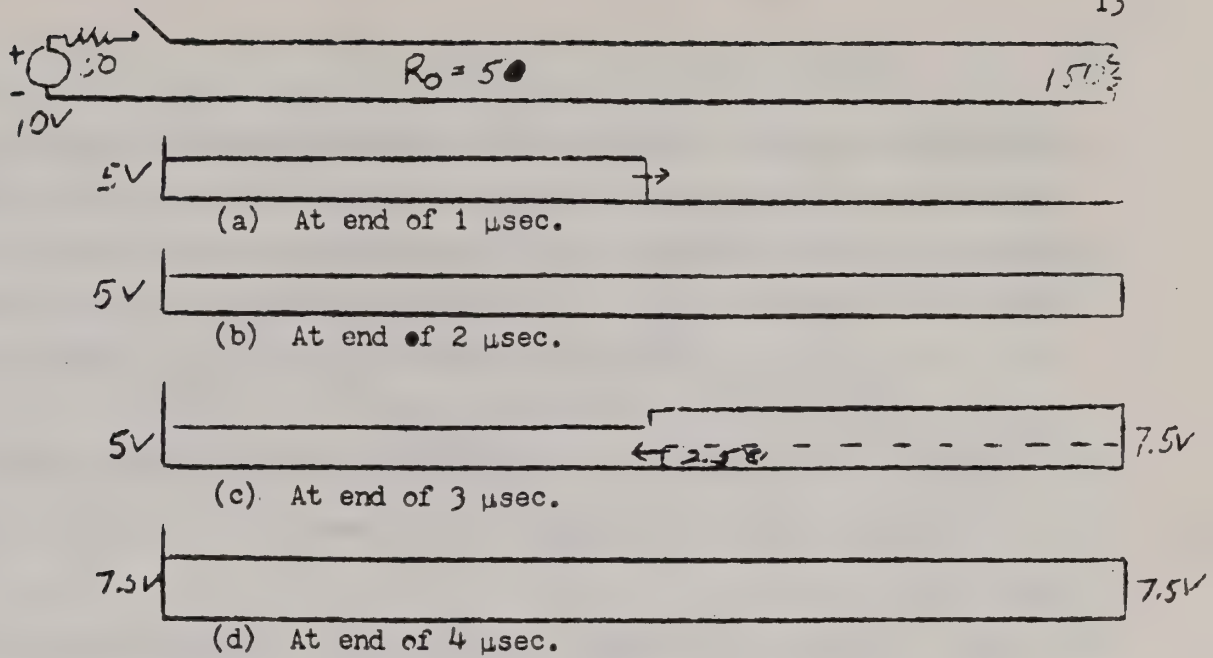


Fig. 8 $e_g = 10V$, $R_g = 50 \text{ ohms}$, $R_L = 150 \text{ ohms}$, for example 2
 $K_{eg} = 0$ and $K_{eL} = +0.5$

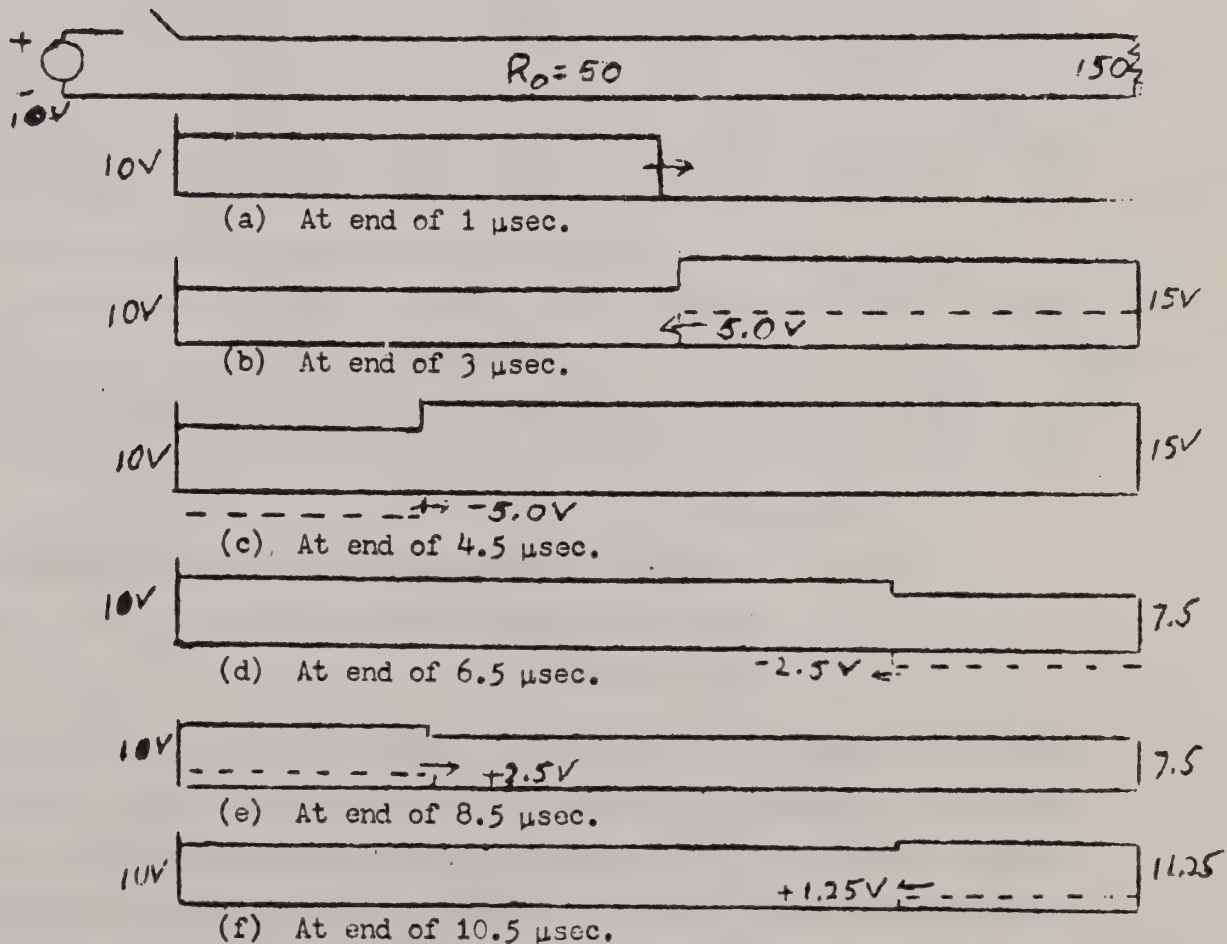


Fig. 9 $e_g = 10V$, $R_g = 0$, $R_L = 150 \text{ ohms}$, for example 3
 $K_{eg} = -1$ and $K_{eL} = +0.5$

Example 3. Suppose the conditions are the same as in example 2 except $R_g = 0$. In this case the voltage reflection coefficient at the source becomes -1.0 . The load reflection coefficient remains at $+0.5$. Now events are pictured in Fig. 9. Graph (b) shows the first reflection of $+ .5 \times 10 = 5$ volts travelling toward the source. At the source -5 volts are reflected and travel toward the load. Graph (d) shows the second reflection of -2.5 volts travelling toward the source where $+2.5$ volts are reflected. The third reflection results in $+1.25$ volts travelling toward the load. Thus the reflections are dying out and the steady state voltage of the entire line becomes 10 volts.

Example 4. Suppose the conditions are the same as in example 1 except the 10 volt battery (step voltage) is replaced by a 20 volt pulse of $1 \mu\text{sec}$. duration. Since $R_L = R_0$ this pulse which will reach the load in $2 \mu\text{seconds}$, will be dissipated entirely in the load resistor, no reflection.

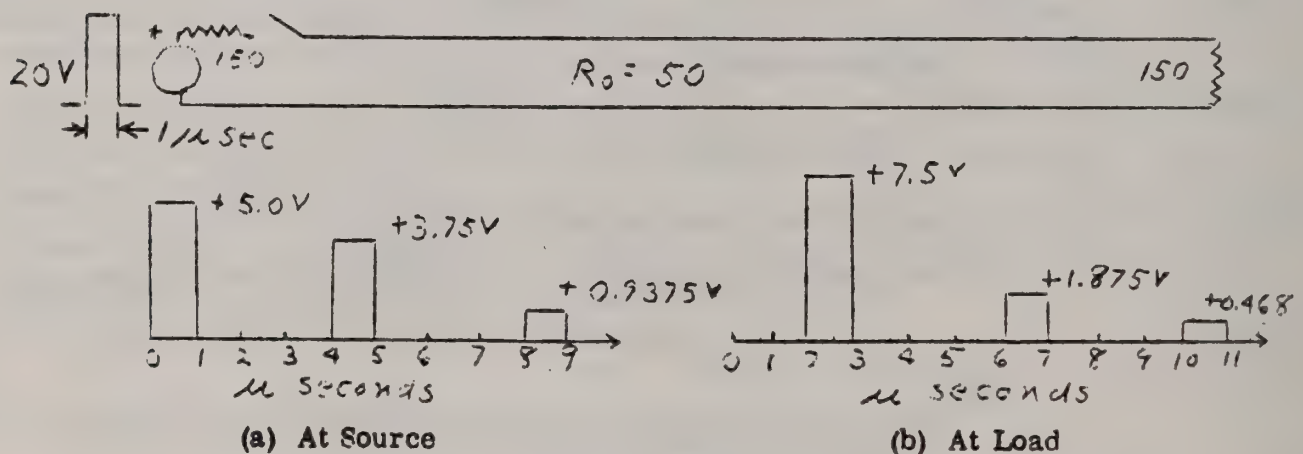


Fig. 10 For each reflection the incident and reflected voltages add to give the total voltage as shown. $K_{eg} = K_{eL} = +.5$ for example 5

Example 5. Suppose the conditions are the same as example 1 except the load resistance is 150 ohms and the source resistance is also 150 ohms. The voltage reflection coefficients at the load and at the source are both equal to 0.5. A $1 \mu\text{sec}$ pulse of 5 volts $[= 50 \times 20 / (150 + 50)]$ travels toward the load and reaches the load in $2 \mu\text{seconds}$. It is reflected at the load as a $+2.5$ volt pulse which travels toward and reaches the source end in another $2 \mu\text{seconds}$. It is reflected at the source as 1.25 volts travels toward the load and reaches the load in an

additional 2 μsec and so on. Figure 10 represents the total voltage source and load for the first several microseconds.

Example 6 Lumped element transmission lines are often used in radar equipment to provide high voltage pulses of short duration for modulation. Figure 11 shows a lumped constant line that is initially charged to 5 KV. Let us determine the load voltage when the switch is closed.

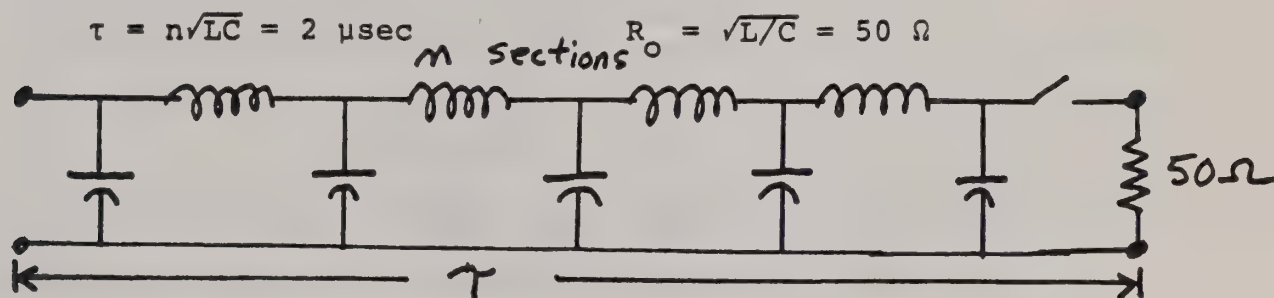


Fig. 11

To determine the initial load voltage at $t = 0+$ we consider the line to exhibit at the load terminals a Thevenin source of 5 KV in series with a 50Ω resistance. Hence the initial load voltage is 2.5 KV. The line voltage has thus dropped by 2.5 KV at the load end. This disturbance then propagates down the line toward the open end discharging line capacitance by 2.5 KV as it moves. Upon reflection from the open end ($K = +1$) the -2.5 KV step proceeds back toward the load dropping the line voltage to zero as it goes. Since the line is $\tau = 2 \mu\text{sec}$ long the resultant load voltage is shown in Fig. 12.

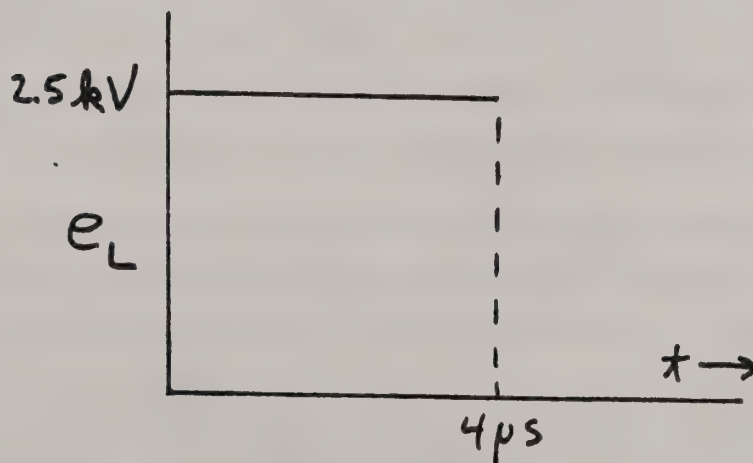


Fig. 12

Example 7 Suppose the system shown in Fig. 13 has reached a steady state condition when at $t=0$ the switch is opened. Let us determine the load voltage e_L as a function of time.

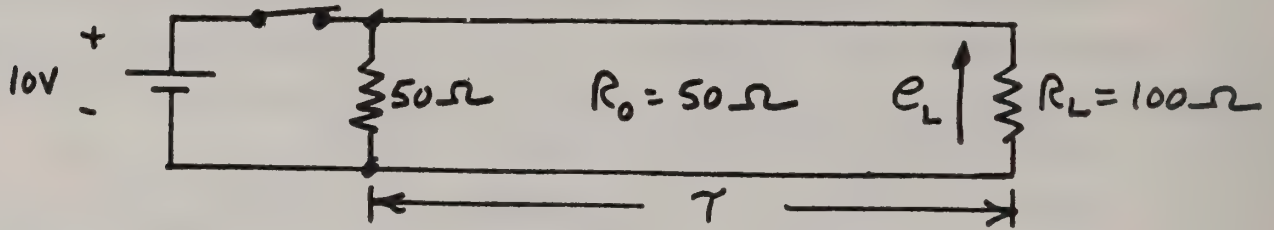


Fig. 13

at $t=0^-$, $i_{IN} = \frac{10}{50} + \frac{10}{100} = \frac{30}{100}$ a. Hence at $t=0^+$ we consider the circuit of Fig. 14 wherein a step current of $-\frac{30}{100}$ a is

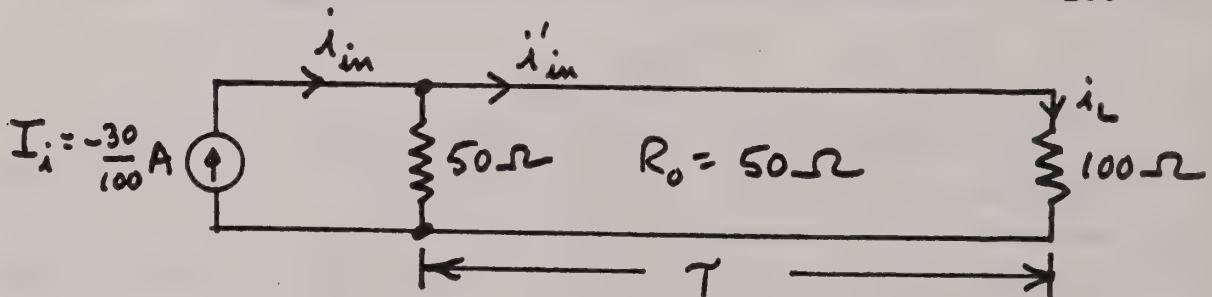


Fig. 14

applied. Since $R_0 = 50\Omega$ this current divides equally at the input resulting in $i'_{IN} = -\frac{30}{200}$ a. After τ seconds this current reaches the 100Ω load where it sees $K_{IL} = -\frac{1}{3}$. A reflected current step of $+\frac{10}{200}$ a then proceeds toward the generator end where it is absorbed τ seconds later. The resultant load current and voltage are shown in Fig. 15 and 16.

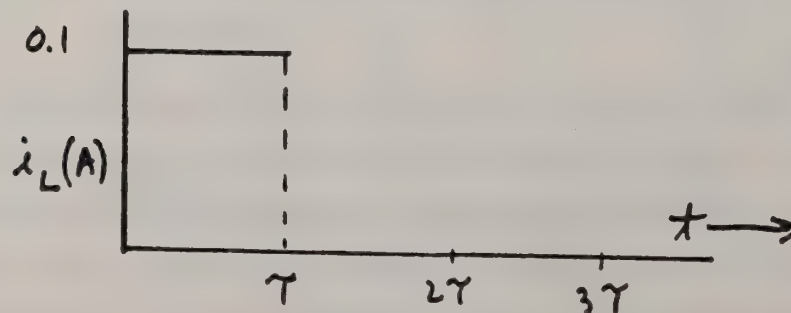


Fig. 15

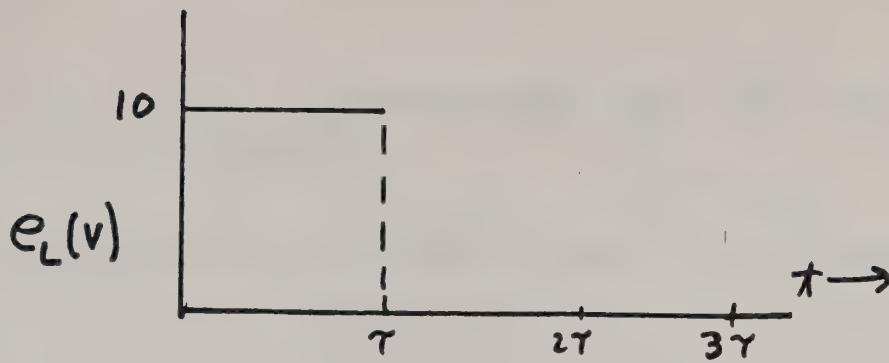


Fig. 16

5. Lines with other than resistive loads

Under transient excitation lines terminated in inductance or capacitance will have reflection coefficients that vary with time. (Note that impedance is a steady state sinusoidal concept and has no application here.) For example, to a voltage step, an inductor looks initially like an open circuit while as $t \rightarrow \infty$ it becomes a short. Hence the voltages and currents on non-resistively terminated lines will vary not only due to time delays introduced by the line but also due to the finite time required to establish a current in an inductor or a voltage across a capacitor.

Consider the circuit of Fig. 17. Let us determine the load reflection coefficient, load voltage and input voltage for this circuit after the switch is closed.

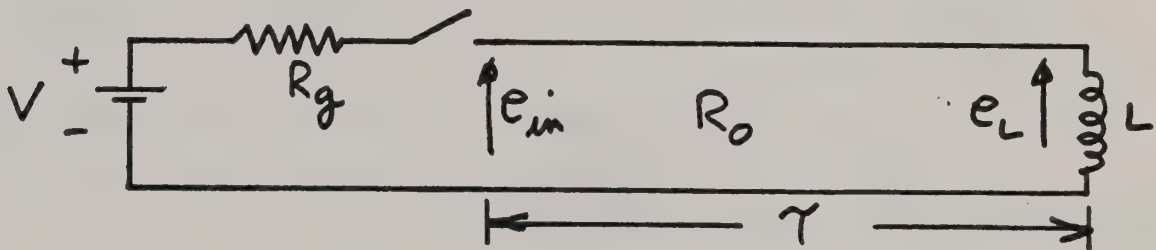


Fig. 17

To find the reflection coefficient we proceed as in the resistive case by expressing the load voltage and current as constrained by the line equations. At $z = 0$,

$$e_L = e_1 + e_2 \quad 5$$

$$i_L = e_1/R_0 - e_2/R_0 \quad 6$$

Also the load itself dictates that

$$e_L = L \frac{di_L}{dt} \quad 7$$

Combining Eqs. 5, 6 and 7 results in the differential equation

$$e_1 + e_2 = \frac{L}{R_0} \left(\frac{\partial e_1}{\partial t} - \frac{\partial e_2}{\partial t} \right) \quad 8$$

However, at $t = 0^+$

$$\frac{\partial e_1}{\partial t} = 0 \quad 9$$

which simplifies Eq. 8, resulting in

$$\frac{L}{R_0} \frac{\partial e_2}{\partial t} + e_2 + e_1 = 0 \quad 10$$

with e_1 constant in time. This equation has the solution

$$e_1 + e_2 = e_0 e^{-R_0 t/L} \quad 11$$

It is convenient to measure time from the arrival of the incident wave at the load, i.e., the switch is closed at $t = -\tau$ seconds. Hence, at $t = 0^+$ the load appears as an open circuit and at $z = 0$

$$e_2 = e_1 \quad 12$$

Equation 11 then yields

$$e_0 = 2e_1 \quad 13$$

Substituting Eq. 13 into Eq. 11 allows the determination of the reflection coefficient at the load as a function of time, viz,

$$\rho_L = \frac{e_2}{e_1} = 2e^{-tR_0/L} - 1 \quad 14$$

Figure 18 shows Eq. 14 as well as the load and input voltages as a function of time.

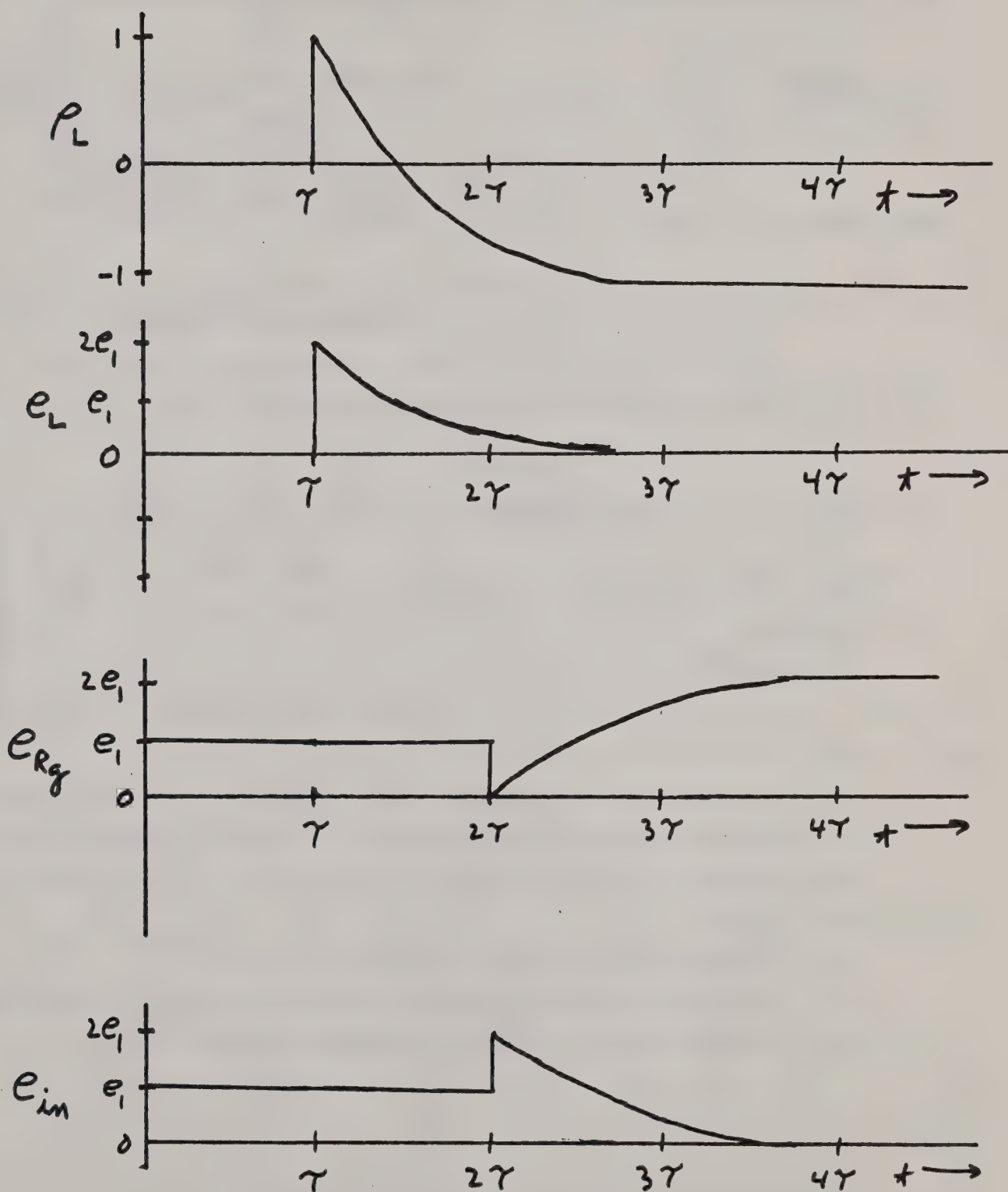
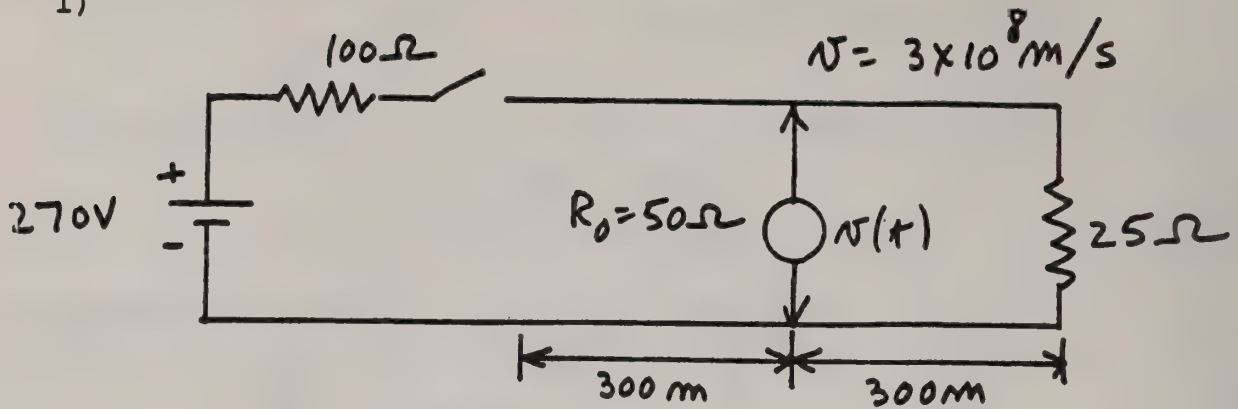


Fig. 18

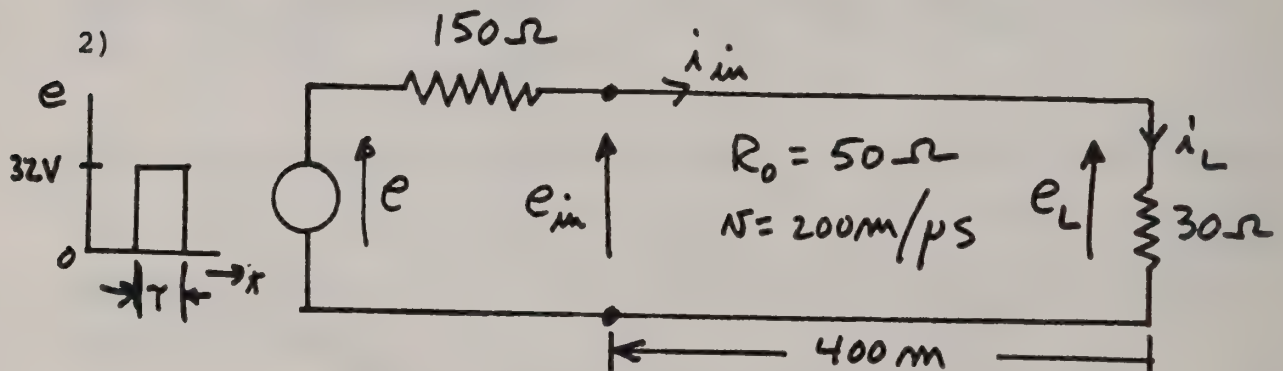
Problems

1)



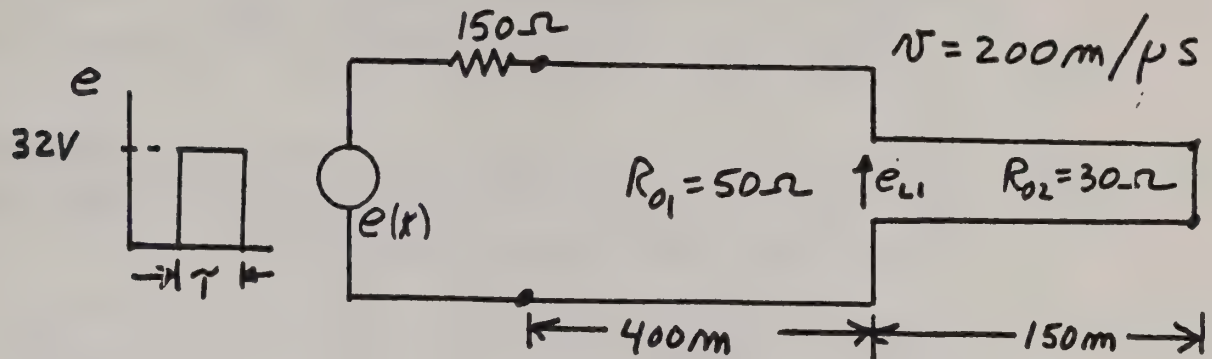
- Determine and plot the voltmeter reading $v(t)$ for the period $t = 0$ to $10 \mu\text{sec}$.
- What is the final voltage across R_L ?

2)

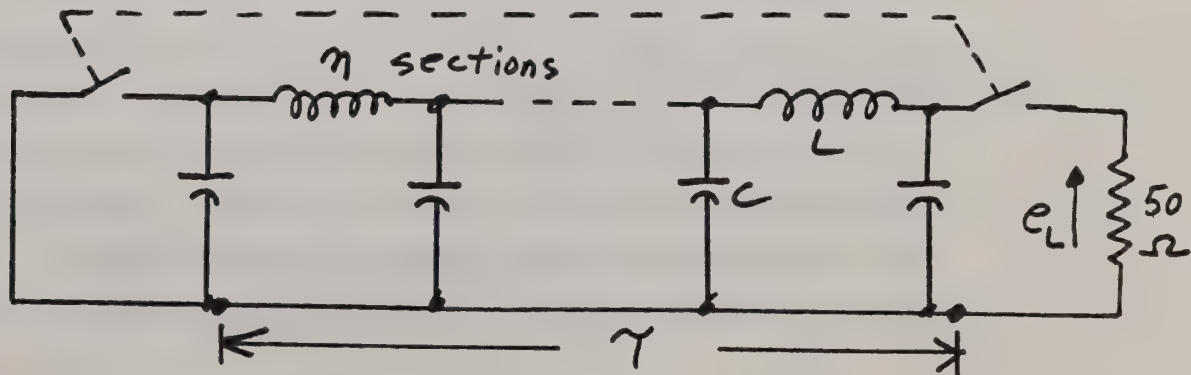


- Determine and plot the input voltage $e_{IN}(t)$ and the input current $i_{IN}(t)$ for the period $t = 0$ to $10 \mu\text{sec}$ if $\tau = 1 \mu\text{sec}$.
- Repeat (a) for the load voltage $e_L(t)$ and the load current $i_L(t)$.
- Repeat parts (a) and (b) for $\tau = 6 \mu\text{sec}$.
- Determine and plot the voltage and current at the midpoint of the line for $0 \leq t \leq 10 \mu\text{sec}$ for $\tau = 1 \mu\text{sec}$.

- 3) Determine and plot $e_{L_1}(t)$ for the period $0 \leq t \leq 10$ sec for $\tau = 1 \mu\text{sec}$.

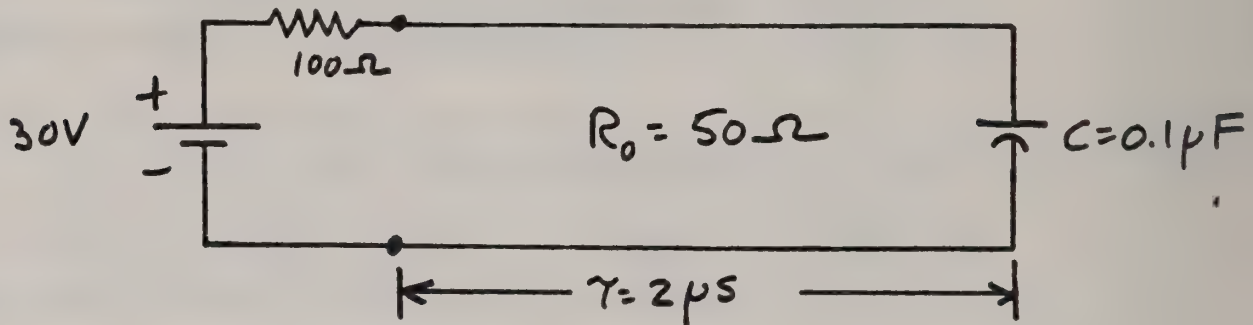


- 4) Find the voltage as a function of time across the 50Ω resistor in Example 7, Fig. 13.
- 5) Assume in Example 7, Fig. 13 that the switch is open and the line initially uncharged. Find and plot $v_L(t)$ for $t \geq 0$.
- 6) A common dielectric for insulating coaxial lines is polyethylene with $\epsilon_r = 2.25$. Show that $v = 200 \text{ m}/\mu\text{sec}$.
- 7) Explain how an oscilloscope, a pulse generator and a precision variable resistor can be used to determine the characteristic resistance of a lossless line.
- 8) A typical coaxial line is designated as RG-8A/U. For this line $R_0 = 50\Omega$ and $v = 200 \text{ m}/\mu\text{sec}$. Find the per meter values of L and C .
- 9) Show in Example 6, Fig. 11 that indeed the delay time $\tau = n\sqrt{LC}$ where L and C are the values of the lumped constants. Determine the values of L and C for this example.
- 10) Suppose Fig. 11, Example 6 is modified as shown.

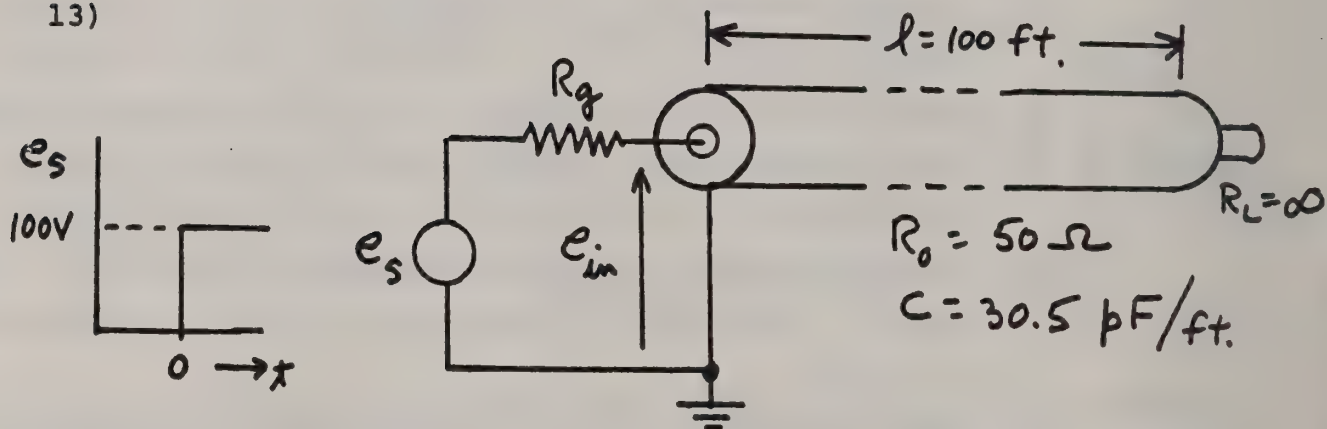


The line is again initially charged to 5 KV. Switches S_1 and S_2 are closed simultaneously at $t=0$. Find $e_L(t)$.

- 11) Determine e_{in} in Section 5, Fig. 17 if $R_0 = 50\Omega$, $R_g = 100\Omega$ and $V = 30$ volts.
- (12) Find and plot the reflection coefficient, load voltage and input voltage as a function of time for the line shown below.



13)



Determine and plot $e_{IN}(t)$ at the input of an open circuit 100 foot length of RG-8A/U coax if $R_g = 1000\Omega$. Find the value of an equivalent capacitor with which the line could be replaced to give essentially the same waveform. Compare the value of this capacitor with the total line capacitance. Replot $e_{IN}(t)$ when $R_g = 50\Omega$. (Hint: $\sum_{n=0}^{\infty} \rho^n = \frac{1-\rho^n}{1-\rho}$) Ans: $C = 2940\text{ pf}$

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
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TECHNICAL SUPPORT PACKAGE

on

DESIGNING HIGH-FREQUENCY INDUCTORS

for January 1987

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Inventor(s): C. W. T. McLyman and
A. P. Wagner

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JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

FEBRUARY 1987

Designing High-Frequency Inductors

Procedures for both ferrite-core and Molypermalloy-core inductors are detailed.

A report presents design procedures for two types of high-frequency inductors: those with iron or ferrite cores and lumped airgaps and those with Molypermalloy powder cores and distributed airgaps. These components, which carry no direct current, are widely used in ferroresonant power regulators, pulse-width-modulation inverters, and other power-processing equipment. They are designed more easily than are dc inductors, using procedures like those for designing transformers. With no dc-generated flux in the core, the design calculations are straightforward. Essentially, the high-frequency ac inductor is designed to support the applied voltage without saturating the core, then the airgap for the proper inductance is calculated.

The report lays the groundwork for the procedures by discussing the two major effects of the inductor airgaps; namely, the increases in the inductance and the power loss due to the fringing flux. For design purposes, the airgap length should be minimized to minimize the fringing flux, thereby maximizing the efficiency. To keep the gap small, the number of turns of wire is kept to a minimum. The number of turns is also minimized to operate the core at the highest possible magnetic-flux density without saturating the core or generating excessive power loss in the core.

The airgap, power handling, and other considerations are represented in the equations for inductor design. The report gives the equations and an 11-step procedure for designing an inductor with a lumped airgap.

Distributed-airgap inductors reduce the losses caused by fringing fluxes. The report presents a design procedure for a toroidal inductor of this type. This procedure is even simpler than that for a lumped-airgap type.

The report concludes with a comparison of the losses among three inductors designed to operate at 20 kHz with a 90-V square wave and a 14-A peak-to-peak triangular wave. Two of the inductors had lumped-airgap cores, and one had a distributed-airgap Molypermalloy-powder core. The temperature rise in the distributed-gap core was less than that of the other two.

This work was done by Colonel W. T. McLyman and Albert P. Wagner of Caltech for NASA's Jet Propulsion Laboratory.

NPO-16553

DESIGNING HIGH FREQUENCY AC INDUCTORS USING FERRITE AND MOLYPERMALLOY POWDER CORES (MPP)

A. INTRODUCTION

Ferro-resonant power regulators, PWM inverters and other power processing equipment in use today include high frequency ac inductors which carry no dc current component. AC inductors may be designed more easily than dc inductors by using a design procedure that is quite similar to the design of a transformer. With no dc-generated flux in the core the design calculations are straight forward. The ac inductor is designed to support the applied voltage without saturating the core. Then the air gap length is calculated for the proper inductance.

An important consideration in designing all high frequency ac inductors is to minimize the fringing flux. The fringing flux can be minimized with the proper selection of the core, the flux density and the gap length.

B. APPARENT POWER

By the authors' definition the "apparent power" P_t of an inductor is the VA of the inductor, that is, the product of the excitation voltage and the current through the inductor:

$$P_t = V_{rms} I_{rms} \text{ [volt amps]}$$

Equation (1)

When a square wave voltage is applied to an inductor with a negligible winding resistance the current has a triangular wave form. The rms value of that current is:

$$I_{rms} = 0.577 I_{pk} \quad [\text{amps}]$$

Equation (2)

When a sine wave voltage is applied the current also has a sine wave form. The rms value of that current is:

$$I_{rms} = 0.707 I_{pk} \quad [\text{amps}]$$

Equation (3)

C. INDUCTANCE EQUATION

The inductance of an iron-core inductor having an air gap may be expressed as:

$$L = \frac{0.4 \pi N^2 A_c 10^{-8}}{l_g + \frac{MPL}{\mu_r}} \quad [\text{henrys}]$$

Equation (4)

Inductance is inversely dependent on the effective length of the magnetic path which is the sum of the air gap length (l_g) and the ratio of the core mean length to relative permeability (MPL / μ_r), i.e., $l_g + MPL / \mu_r$.

The relative permeability (μ_r) is normally very large compared to the magnetic path length (MPL) and therefore the second term of the

denominator of equation (4) becomes very small and variations in the relative permeability do not substantially effect the total effective magnetic path length or the inductance.

The inductance equation then reduces to:

$$L \approx \frac{0.4 \pi N^2 A_c 10^{-8}}{l_g} \quad [\text{henrys}]$$

Equation (5)

D. FRINGING FLUX

Final determination of the gap dimension requires consideration of the effect of fringing flux, which is a function of gap length, the shape of the pole faces and the shape, size and location of the winding. Its net effect is to make the effective air gap shorter than its physical dimension. Fig. 1. illustrates the fringing flux occurring in a gapped inductor.

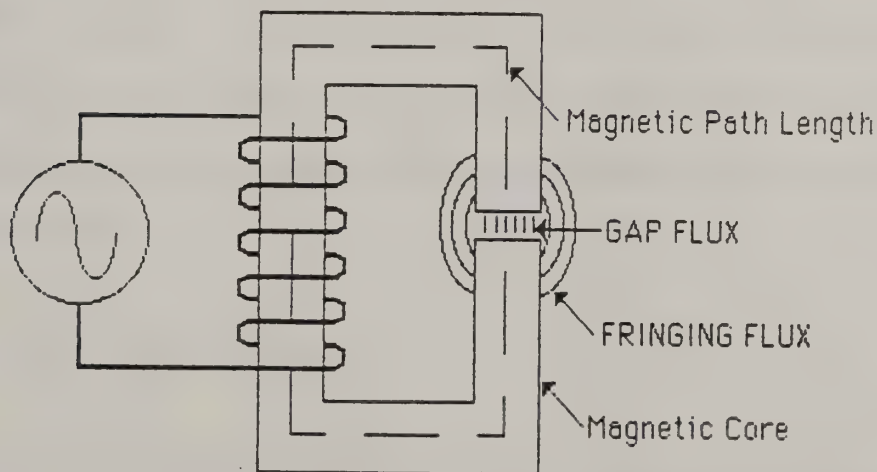


Fig. 1 Gapped core with fringing flux

Fringing flux decreases the total reluctance of the magnetic path and

therefore increases the inductance by a factor F to a value greater than that calculated from equation (4). Fringing flux is a larger percentage of the total for longer gaps. The fringing flux factor is:

$$F = \left(1 + \frac{l_g}{\sqrt{A_c}} \ln \frac{2G}{l_g} \right)$$

Equation (6)

Inductance L computed in equation (5) does not include the effect of fringing flux. The value of inductance L corrected for fringing flux is:

$$L = \frac{0.4 \pi N^2 A_c F 10^{-8}}{l_g} \quad \text{[henrys]}$$

Equation (7)

E. TRANSFORMER LOSSES

The losses in a normal transformer result from two components, copper loss (P_{cu}) and iron loss (P_{fe}). Normally when these losses are equal the transformer has reached its highest efficiency. This is only true when the gap is zero.

F. EDDY-CURRENT LOSSES

Any iron core magnetic component with a large air gap inherently operates with a power loss associated with the air gap. The loss does not occur in the air gap itself, but is caused by eddy currents which are due to magnetic flux fringing around the gap and re-entering the core and/or the

winding in a direction of high loss. As the air gap increases, the fringing flux increases which in turn causes the eddy loss to increase. The eddy current loss can be generated in either the copper winding (especially if foil is used) or in the core.

Normally high frequency ac inductors are not designed using laminations or tape cores because of their high eddy current loss. Eddy currents

generated in the core can be minimized by the use of high resistivity ferrite material or molypermalloy power cores (MPP). Summarizing--the total losses in a magnetic component with an air gap are made up of three components:

(1) Copper loss, P_{cu}

(2) Iron loss, P_{fe}

(3) Gap loss, P_g

6. POWER HANDLING EQUATIONS FOR AC INDUCTORS

The same equations used for designing power transformers are used for designing ac power inductors. There are two methods used in designing them.* The first method uses the area product approach. The area product, A_p , represents the power handling ability of the core. The area product is the iron cross section, A_c times the window area, W_a

$$A_p = A_c W_a \quad [\text{cm}^4]$$

Equation (8)

* References [1] [2]

The power handling capability of a core is related to its area product by an equation which may be stated as:

$$A_p = \frac{P_t \cdot 10^4}{K_u K_f B_m f J} \quad [\text{cm}^4]$$

Equation (9)

The second method uses the core geometry factor K_g approach. Like the first method the core geometry K_g is also a measure of the power handling ability of the core. The core geometry is equal to the area product, A_p , times the iron cross section A_c times the window utilization K_u divided by the mean length turn MLT.

$$K_g = \frac{W_a A_c^2 K_u}{\text{MLT}} \quad [\text{cm}^5]$$

Equation (10)

The power handling capability of a core is related to its core geometry by the equation:

$$K_g = \frac{P_t}{0.145 f^2 B_m^2 K_f^2 \alpha 10^{-4}} \quad [\text{cm}^5]$$

Equation (11)

Either design equation will produce a magnetic component. The advantage the core geometry method has over the area product method is the expression α in the denominator. The expression α is the regulation or copper loss of the transformer or inductor in percent. For example if α is equal to 1 then the transformer will have a 1 percent regulation if α is equal to 10 then the transformer will have a 10 percent regulation. Use of the area product equation will result in the same design as using the core geometry equation but will require several iterations. This is because the area product equation uses a fixed current density factor, J , which is usually approximated in the first iteration.

H. DESIGNING GAPPED AC INDUCTORS

It is important when designing an ac inductor that the air gap length be minimized. A minimum air gap provides maximum efficiency by reducing the fringing flux losses to a minimum. To keep the gap at a minimum the turns are kept to a minimum. The number of turns is also minimized to operate the core at the highest possible flux density (B_m) without saturating the core or generating excessive core loss.

A core is used that has a high ratio between iron area A_c and window area W_a .

$$A_c \gg W_a$$

The reason for designing with this A_c / W_a ratio follows. Two cores A and B having the same area product A_p or power handling capability but

different window areas are shown here:

CORE A	CORE B
$W_a = 1$	$W_a = 2$
$A_c = 2$	$A_c = 1$
<hr/>	
$A_p = 2$	$A_p = 2$

From equations (12) and (13) it can be shown that for the same inductance core A requires half the number of turns and half the air gap length

resulting in reduced gap losses and reduced copper losses.

The ac inductor is designed to support the applied voltage using Faraday's law:

$$N = \frac{V \cdot 10^4}{K_f f A_c B_m} \text{ [turns]}$$

Equation (12)

After the turns have been established the required gap length is calculated.

$$l_g = \frac{0.4 \pi N^2 A_c 10^{-8}}{L} \text{ [cm]}$$

Equation (13)

After the air gap has been calculated the fringing flux must be calculated using equation (6) in order to make corrections necessary to achieve the required inductance. The error introduced by the fringing flux can be

corrected in two ways. The turns can be reduced or the air gap can be increased. Normally the turns are set by the flux density so the air gap has to be increased. After the fringing flux has been calculated the original air gap dimension is multiplied by the fringing flux factor as shown in equation 14.

$$l_g = l_g F \quad [\text{cm}]$$

Equation (14)

With this new air gap the resulting inductance is calculated using equation 15.

$$L' = \frac{0.4 \pi N^2 A_c 10^{-8}}{l_g} \quad [\text{henrys}]$$

Equation (15)

Now that the new inductance has been calculated the fringing flux factor (F') for the new air gap is calculated using equation 6. Now the fringing flux factor is multiplied by the inductance found in equation 15 as shown in equation 16.

$$L'' = L' F' \quad [\text{henrys}]$$

Equation (16)

I. DESIGN PROCEDURE FOR GAPPED INDUCTOR - RESUME'

No. 1

Calculate the apparent power (P_t) using equation (1) and the appropriate current in equation (2) or (3).

No. 2

6053

16553

Calculate the area product, A_p , using equation (9) and an estimated value of J or calculate the core geometry, K_g , using equation (11)

No. 3

From ferrite core tables select a core having an A_p or a K_g^* just above the value calculated in step No. 2.

No. 4

Now calculate the number of required turns using equation (12).

* See references for sources of core tables

No. 5

Calculate the allowable current density, J , using equation (18) using the value of A_p given in the table for the selected core. If the value of J is considerably different from the estimated J used in step 2 then repeat the design from step 2 using the new J .

No. 6

Calculate the bare wire area, $A_w(B)$, by dividing the current density J into the rms current.

No. 7

Calculate the required air gap, l_g using equation (13).

No. 8

Calculate the fringing flux, F , using equation (6).

No. 9

Calculate the new gap dimension, l_g' , using equation (14).

No. 10

Calculate the inductance, L' , using equation (15) and the new air gap dimension, l_g' , found in step (9).

No. 11

Calculate the final inductance, L'' , using equation (16) and the fringing flux found in step 8.

In most cases correction of the air gap length using the fringing flux

calculations just discribed will result in an inductance close to the design value. When the measured inductance is outside the tolerances allowable then the air gap must be adjusted to compensate for core tolerances and inaccuracies in the fringing flux equation. In general it is not adviseable to use an air gap smaller than 5 mils in length because then any change in permeability of the magnetic material will have an influence on the inductance see equation (4) and it is too difficult to control the banding pressures because of the small gap. This precaution becomes important when a production run is to be made of a particular design and the units must fall within a tolerance band. It is also important to standardize and control the banding pressure documenting the banding conditions for furture runs.

J. DESIGNING THE TOROIDAL AC INDUCTOR WITH A DISTRIBUTED GAP CORE

A way to reduce the losses caused by the fringing flux is to use a core with a distributed air gap such as a molypermalloy power core (MPP).

Designing an AC inductor with a distributed air gap is easier than designing one with a lumped air gap. The power handling equations (9) and (11) are still valid to size the core. After the core size has been determined from these equations a core must be selected from the group with the right permeability which is determined by using equation (17). Normally there are about 10 different permeabilities to choose from for a given size. The permeabilities range from a low of 14u to a high of 550u.

$$\mu_r = \frac{B_m \text{ MPL } 10^4}{0.4 \pi W_a J K_u}$$

For J in equation (17) enter the value used in calculating A_p from equation (9). If the core size was determined using equation (11) then calculate J from equation (18) below using the value of A_p given in a table listing that core size.

$$J = \frac{P_t \cdot 10^4}{K_f K_u B_m f A_p} \quad [\text{amps /cm sq}]$$

Equation (18)

Select a core from the group that has the closest permeability to the one calculated. Then calculate the number of turns for the required inductance. This is done using equation (19).

$$N = 1000 \sqrt{\frac{L_{\text{new}}}{L_{1000}}} \quad [\text{turns}]$$

Equation (19)

L_{new} = required inductance, millihenrys

L_{1000} = millihenrys / 1000 turns (from core table)

K. COMPARISON DESIGN

Three ac inductor design comparisons were made using two different ferrite core configurations, a pot core, EC core, and a toroidal MPP core. The cores' electrical and mechanical parameters along with the test data are shown in the comparison table below. The purpose of the tests was to show that in each configuration the power loss due to the air gap was significant. The approximate copper and core losses were measured separately. The total loss was then measured and the gap loss calculated

by subtracting the copper and iron losses from the total loss.

Design and Test Conditions

The inductance was approximately 140 micro-henrys in each case and the inductors were operated under the following conditions.

1. frequency, 20kHz
2. applied voltage waveform, square
3. voltage, 90V
4. triangular current, 14 Amps p-p

Copper Loss

In each case the temperature rise due to the copper loss was determined using a DC current equivalent to the ac rms current of equation (2). The approximate temperature rise data due to copper loss alone can be found in the comparison table under P_{cu} .

Core loss

For the two ferrite cores the temperature rise due to core loss alone was evaluated by reducing the air gap to zero. The AC voltage was applied directly to the inductor. Without the air gap the magnetizing current is at least two orders of magnitude below the normal magnetizing current and is so low that the copper loss is negligible. The core loss depends only on the applied voltage and is independent of the current amplitude. The approximate temperature rise data due to core loss alone can be found in the comparison table under P_{fe} .

Gap Loss, Core Loss and Copper Loss

The combined temperature rise due to all three losses, gap, core and copper, was measured with each inductor operating with its designed air gap. The losses are found in the comparison table under P_j . The temperature rise in the distributed gap MPP core was much less than in the two ferrite cores with their relatively large air gap.

The approximate gap losses were then calculated and entered in the table.

References *

- [1] Colonel McLyman, Transformer and Inductor Design Handbook, (New York: Marcel Dekker, 1978).
- [2] Colonel McLyman, Magnetic Core Selection for Transformers and Inductor, (New York: Marcel Dekker, 1982).

*PLEASE OBTAIN REFERENCES FROM THE SOURCE LISTED.



TYPE	COMPARISON TABLE		
	POT- CORE FERRITE	EC-CORE FERRITE	TOROID MPP
1 CORE NO.	3019	EC-35	55551-A2
2 $K_g \text{ cm}^5$	0.092	0.067	0.107
3 $A_p \text{ cm}^4$	1.017	1.240	1.863
4 $W_a \text{ cm}^2$	0.747	1.48	2.773...
5 $A_c \text{ cm}^2$	1.60 avg	0.84	0.672
6 $MP_L \text{ cm}$	4.5	7.7	8.15
7 $A_t \text{ cm}^2$	32.0	42.8	48.0
8 $G \text{ cm}$	1.3	2.38	NA
9 $l_c \text{ cm}$	0.254	0.950	NA
10 F	1.47	2.670	NA
11 MAT. PERM	2300	2300	NA
12 EFF. PERM	17.5	7.90	14
13 R_{cu} RESISTANCE	0.0499	0.0835	0.0740
14 B_m tesla	0.30	0.20	0.167
15 P_{cu} TEMP. RISE	10 C	10 C	10 C
16 P_{fe} TEMP. RISE	10 C	10 C	NA
17 P_j TEMP RISE	70 C	95 C	55 C
18 WIRE NO. AWG	4 # 24	4 # 24	5 # 24
19 TURNS	34	67	100



SYMBOLS

α ,	regulation (%)	μ_r ,	relative permeability
A_c ,	iron cross-section (cm sq)	VA ,	volt-amps
A_p ,	area product (cm 4th)	w_a ,	window area (cm sq)
A_t ,	surface area (cm sq)		
B_m ,	ac flux density (tesla)		
F ,	fringing flux		
f ,	operating frequency (Hz)		
G ,	window length (cm)		
h ,	separation between coil and core (cm)		
J ,	current density (amps / cm sq)		
K_f ,	waveform factor		
K_g ,	core geometry (cm 5th)		
K_u ,	window utilization		
L ,	inductance (henrys)		
l_g ,	gap length (cm)		
MLT ,	mean length turn (cm)		
MPL ,	magnetic path length (cm)		
N ,	number of turns		
P_{cu} ,	copper loss (watts)		
P_{fe} ,	iron loss (watts)		
P_g ,	gap loss (watts)		
P_j ,	combine loss (watts)		
P_t ,	apparent power (volt amps)		

Transmission Line Theory. (General)

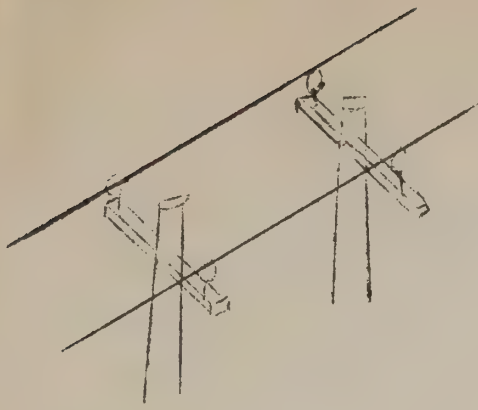
1. Introductory

The transmission lines treated in these notes consists of two conductors, usually copper or aluminum, arranged in various configurations some of which are illustrated symbolically by Fig.1. The purpose of a transmission line is to guide electrical energy in some form, such as 60 cycle energy for home and commercial purposes or modulated energy in the form of telephone conversation and others, from one location to another. The conductors of the transmission line do not transmit the energy but serve only as guides for the energy which is transmitted in the electromagnetic field surrounding the conductors. This is the only concept that satisfies the Maxwell field equations and agrees with other systems of energy transmission such as hollow wave guides where only one conductor exists and radio transmission where there are no conductors.*

The mathematical treatment and physical behavior of a transmission line depends upon whether the line is electrically short or long. Actual physical length and electrical length are two different quantities when deciding whether a line is short or long. A transmission line is 100 meters long physically. At 60 cycles per second it is very short electrically. At 10^8 cycles per second it is very long electrically. Electrical length is expressed in terms of wavelength for a particular operating frequency. Wave length is the distance for which a voltage, or current, undergoes a 360° phase shift from one end of the line to the other. One wave length at 60 cps is 3100 miles. Whereas it is only 3 meters at 10^8 cps.

A transmission line is a linear circuit consisting of resistance R, inductance L, capacitance C and leakage conductance G uniformly distributed throughout the length of the line. The resistance and inductance are in series with the line conductors and the capacitance and leakage conductance are in shunt as illustrated in Fig. 2. Because of the distributed nature of these parameters the current and voltage on a line undergo continual changes in magnitude and phase along the line in relation to the current and voltage at some reference point such as the sending end or the receiving end.

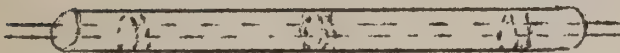
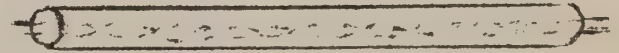
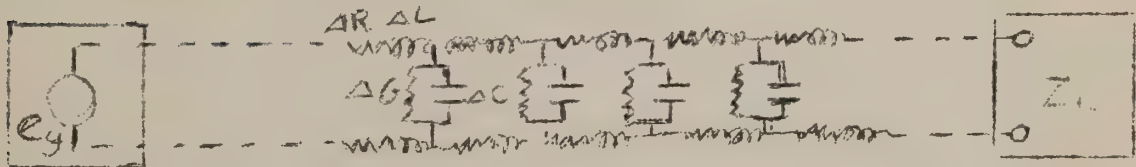
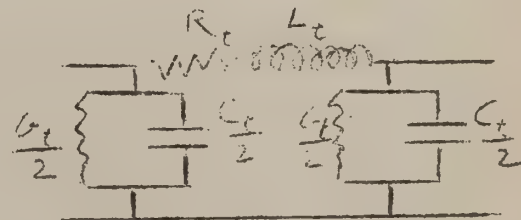
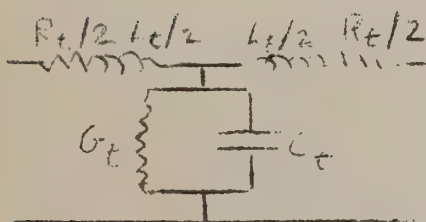
A very short, electrically, length of transmission line may be represented as a T or π network as shown in Fig. 3. For example 50 miles of 60 cycle power line is only .0155 wave lengths long and may therefore be represented as a single T or π network. For a power line only a few miles long it is generally sufficient



a. open wire line



b. Coaxial line

c. shielded pair line
outside shield usually
copper and groundedd. cable type line
sheath may be metallic
some cables contain
several pairs of conductorsFig. 1. Several Types of Transmission LinesFig. 2. A very short piece of transmission line between the source e_g and the load Z_L showing how R, L, C and G are disposed to form a circuit of uniformly distributed parameters.Fig. 3. A very short transmission line represented by a T or π network. The parameters R_t, L_t, C_t , and G_t are for the entire length of line.

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to replace the line by its overall resistance and inductance and neglect its capacitance and leakage conductance.

When a line becomes of the order of .05 wave lengths or longer and there is only one frequency of interest the line may be represented by a single T or π network whose parameters are determined from the open and short circuit impedances of the line. However, open and short circuit impedances must be determined either from measurements or from mathematical theory which takes into consideration the uniform distribution of the line parameters. Thus for a line that cannot be represented by a T or π network without the use of the open and short circuit impedance it becomes simpler to treat the line as a uniformly distributed parameter circuit and derive and solve the differential equation which apply.

2. Derivation of the Differential Equations for the Transmission Line.

In order to arrive at a pair of equations which express the relations for voltage and current at any point on a transmission line it is necessary to first derive the fundamental differential equations for the line. These equations are then solved for voltage and current in terms of the line parameters, the distance to a point in question and the terminal conditions.

The differential equations are developed by applying the e.m.f. and current laws to an infinitesimal portion of the line. Fig. 4 represents an infinitesimal portion of a transmission line. Since the current in one conductor is equal to, but opposite in direction, at any instant of time to the current in the other conductor, all line resistance and inductance of the infinitesimal portion of line is placed in the upper branch as shown.

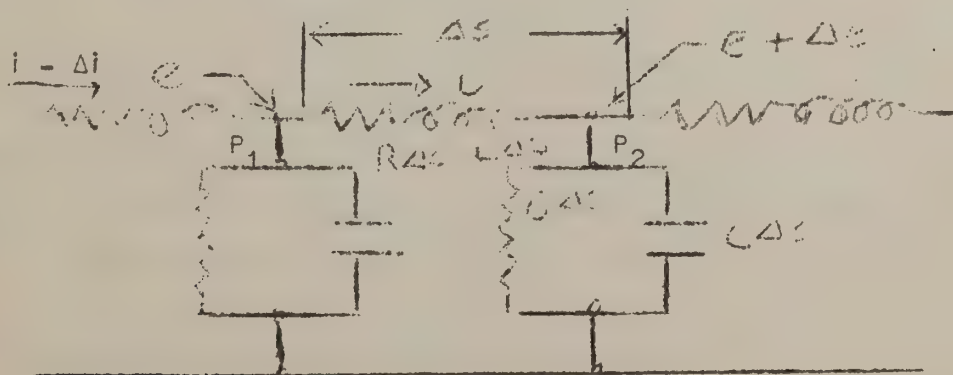


Fig. 4 An infinitesimal length of transmission line. The parameters R, L, C and G are for a unit length of line.

Referring to Fig. 4 the difference in potential between the line conductors at point P_2 is equal to the potential difference e at P_1 plus Δe where Δe is the incremental change in potential difference in going from P_1 to P_2 because of the current i in RA s and LA s. In equation form this becomes

$$e + (Ri + L \frac{\partial i}{\partial t}) \Delta s = e + \Delta e$$

Which gives, as $\Delta s \rightarrow 0$

$$\frac{\partial e}{\partial s} = - (Ri + L \frac{\partial i}{\partial t}) \quad (1)$$

In the infinitesimal portion of line from P_1 to P_2 the current i and potential e are changing with respect to both time and distance, thus the partial derivatives.

In a similar manner as the above the current to the right of P_2 differs from the current between P_1 and P_2 by Δi where Δi is the incremental change in current due to the shunt paths GA s and CA s. In equation form

$$i = i - \Delta i - (G e + C \frac{\partial e}{\partial t}) \Delta s$$

which gives, as $\Delta s \rightarrow 0$

$$\frac{\partial i}{\partial s} = - (G e + C \frac{\partial e}{\partial t}) \quad (2)$$

Equations 1 and 2 are the fundamental differential equations for a transmission line. The equations give the space rate-of-change of e and i respectively, and at each point on the line e and i are changing with respect to time. This indicates that e and i are propagated along the line in a wave pattern. This will become clearer when the differential equations are solved and physical interpretations are given for the resulting solutions.

Solutions of equations 1 and 2 are best carried out by first obtaining two equations, each of which is expressed in terms of a single dependent variable, i.e. one equation containing only e and the other containing only i . This is carried out as follows.

Differentiation of 1 with respect to s gives

$$\frac{\partial^2 e}{\partial s^2} = - (R \frac{\partial i}{\partial s} + L \frac{\partial^2 i}{\partial t \partial s}) \quad (3)$$

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$$1 - \frac{1}{2} = \frac{1}{2}$$

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$$1 - \frac{1}{2} = \frac{1}{2}$$

Differentiation of 2 with respect to t gives

$$\frac{\partial^2 i}{\partial t \partial s} = (G \frac{\partial e}{\partial t} + C \frac{\partial^2 e}{\partial t^2}) \quad (4)$$

Now substitute $\frac{\partial i}{\partial s}$ from equation 2 and $\frac{\partial^2 i}{\partial t \partial s}$ from equation 4 into equations 3 and get

$$\frac{\partial^2 e}{\partial s^2} = RGe + (RC + LG) \frac{\partial e}{\partial t} + LC \frac{\partial^2 e}{\partial t^2} \quad (5)$$

In like manner

$$\frac{\partial^2 i}{\partial s^2} = RGi + (RC + LG) \frac{\partial i}{\partial t} + LC \frac{\partial^2 i}{\partial t^2} \quad (6)$$

These are the second order differential equations that govern the behavior of e and i for the transmission line.

Except for the special case which is the steady state solution when e and i are sinusoidal functions of time the solutions of equations 5 and 6 are quite complicated both mathematically and physically. Hence in order to acquire some physical insight into the nature of e and i before proceeding with steady state sinusoidal solutions equation 5 and 6 will be modified to represent the loss-less line condition, $i, e, R = G = 0$. In this sense they become

$$\frac{\partial^2 e}{\partial s^2} = LC \frac{\partial^2 e}{\partial t^2} \quad (7)$$

$$\frac{\partial^2 i}{\partial s^2} = LC \frac{\partial^2 i}{\partial t^2} \quad (8)$$

3. Travelling Waves.

Equations 7 and 8 are generally known as the wave equations for loss-less transmission lines. Just why these equations portray wave motion may be seen by examining in some detail their steady state solutions when e and i are sinusoidal functions of time. An assumed solution for equation 7 is

$$e = e_1 \cos (\omega t - \beta s) + e_2 \cos (\omega t + \beta s) \quad (9)$$

10

$$f(x) = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots$$

11

$$f(x) = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots$$

The function $f(x) = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots$ is a series of terms that decrease as x increases. The first term is $\frac{1}{x^2}$, the second is $\frac{1}{x^3}$, the third is $\frac{1}{x^4}$, and so on. The series converges for $x > 1$. The sum of the series is $\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots = \frac{1}{x^2} \left(1 + \frac{1}{x} + \frac{1}{x^2} + \dots \right) = \frac{1}{x^2} \left(\frac{1}{1 - \frac{1}{x}} \right) = \frac{1}{x^2} \left(\frac{x}{x-1} \right) = \frac{1}{x(x-1)}$.

$$\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots$$

$$\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots$$

The function $f(x) = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots$ is a series of terms that decrease as x increases. The first term is $\frac{1}{x^2}$, the second is $\frac{1}{x^3}$, the third is $\frac{1}{x^4}$, and so on. The series converges for $x > 1$. The sum of the series is $\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots = \frac{1}{x^2} \left(1 + \frac{1}{x} + \frac{1}{x^2} + \dots \right) = \frac{1}{x^2} \left(\frac{1}{1 - \frac{1}{x}} \right) = \frac{1}{x^2} \left(\frac{x}{x-1} \right) = \frac{1}{x(x-1)}$.

That this is a solution may be shown as follows:

$$\frac{\partial e}{\partial s} = + \beta e_1 \sin (\omega t - \beta s) - \beta e_2 \sin (\omega t + \beta s)$$

$$\frac{\partial^2 e}{\partial s^2} = - \beta^2 e_1 \cos (\omega t - \beta s) + \beta^2 e_2 \cos (\omega t + \beta s)$$

$$= \beta^2 [+ e_1 \cos (\omega t - \beta s) + e_2 \cos (\omega t + \beta s)]$$

Likewise

$$LC \frac{\partial^2 e}{\partial t^2} = - \omega^2 e_1 \cos (\omega t - \beta s) + \omega^2 e_2 \cos (\omega t + \beta s) \quad LC$$

$$= \omega^2 [+ e_1 \cos (\omega t - \beta s) + e_2 \cos (\omega t + \beta s)] \quad LC$$

Then if $\beta = \omega \sqrt{LC}$ equation 9 becomes a solution of equation 7. Now by graphing the first term of equation 9 in Fig. 5 for several instants of time it is seen that this term represents a wave moving in the positive direction of s . In a similar manner the second term of equation 9 represents a wave moving in the negative s direction. For one complete period, i.e. $1/\text{frequency}$, the wave moves a distance called one wave length λ . Thus

$$s_\lambda = \lambda = \frac{2\pi}{\beta} \quad \text{OR} \quad \beta = \frac{2\pi}{\lambda} \quad (10)$$

Now s = velocity of wave multiplied by time or

$$s_\lambda = \lambda = \text{vel.} \cdot t = \text{vel.} / f = \frac{2\pi}{\beta}$$

$$\text{Hence} \quad \text{vel} = \frac{2\pi f}{\beta} \quad (11)$$

$$\text{and} \quad \beta = 2\pi f \sqrt{LC} \quad (12)$$

Consequently velocity = $\frac{1}{\sqrt{LC}} = v_p$ and is called the phase velocity of the wave. For a transmission line with air or vacuum dielectric it may be shown that if the self inductance due to magnetic flux linkages inside the conductors is neglected $1/\sqrt{LC} = c$ the velocity of light in free space. For any lossless line $v_p = c/\sqrt{\epsilon_r}$ where ϵ_r is the dielectric constant of the medium around the line conductors.

Let $f(x) = x^2 + 1$ and $g(x) = x^2 - 1$.

(a) Find $(f+g)(x)$ and $(f-g)(x)$.

(b) Find $(fg)(x)$ and $(f/g)(x)$.

(c) Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

(d) Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Let $f(x) = x^2 + 1$ and $g(x) = x^2 - 1$. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, $(f/g)(x)$, $(f \circ g)(x)$, and $(g \circ f)(x)$.

$$(f+g)(x) = (x^2 + 1) + (x^2 - 1) = 2x^2$$

Let $f(x) = x^2 + 1$ and $g(x) = x^2 - 1$.

$$(f-g)(x) = (x^2 + 1) - (x^2 - 1) = 2$$

$$(fg)(x) = (x^2 + 1)(x^2 - 1) = x^4 - 1$$

Let $f(x) = x^2 + 1$ and $g(x) = x^2 - 1$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

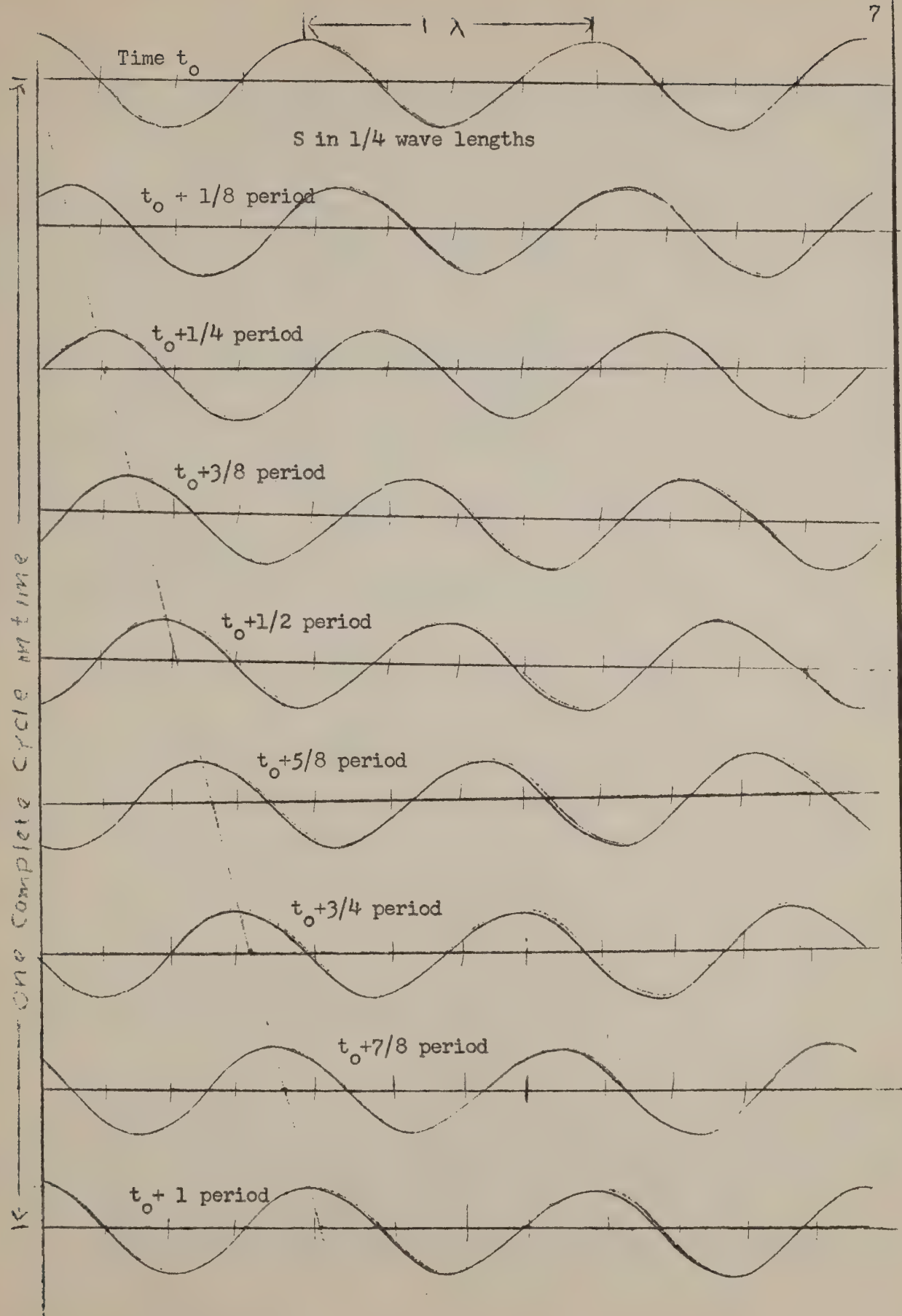


Fig. 5. Wave moves to right 1λ in 1 cycle of time

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Assume now a line so terminated that only the first term of equation 9 exists. This is possible as will be seen later. Suppose two oscilloscopes were connected across the line s meters apart. Both oscilloscopes would show sinusoidal voltage - time waves. However, the voltage-time wave shown on the oscilloscope farthest from the source end would be βs degrees lagging the voltage-time wave shown by the other oscilloscope. Suppose the oscilloscope nearest the source were connected to the line at the instant the voltage was e_a volts and then moved along the line at a velocity v_p . The oscilloscope would continue to show e_a volts. These observations help to show that a travelling wave exists on the line.

When the resistance and leakage conductance are not zero, as is the case for a realizable line, the travelling waves suffer attenuation along the line. This will be shown later when the equations for the general case are discussed.

4. Transient considerations.

Returning now to the basic differential equations namely

$$\frac{\partial e}{\partial s} = -(Ri + L \frac{\partial i}{\partial t})$$

$$\frac{\partial i}{\partial s} = -(Ge + C \frac{\partial e}{\partial t})$$

and setting $R = G = 0$ for the loss less case there results

$$\frac{\partial e}{\partial s} = -L \frac{\partial i}{\partial t} \quad (13)$$

$$\frac{\partial i}{\partial s} = -C \frac{\partial e}{\partial t} \quad (14)$$

A general solution of equation 13 for e is

$$e = e_1 f(t - \frac{s}{v}) + e_2 f(t + \frac{s}{v}) \quad (15)$$

where $v = 1/\sqrt{LC}$ and is the velocity of propagation as was found for the steady state solution when e and i vary sinusoidally with time. It may be shown from fundamental dimension that $1/\sqrt{LC}$ is velocity.

Now examining, in detail the first term of equation 15 it is seen that

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$$\begin{aligned}\frac{\partial e}{\partial s} &= e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \frac{\partial(t - \frac{s}{v})}{\partial s} \\ &= e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \left(-\frac{1}{v}\right)\end{aligned}\quad (16)$$

Assume now the corresponding solution for i , i.e.

$$i = \frac{e_1}{R_o} f(t - \frac{s}{v}) + \frac{e_2}{R_o} f(t + \frac{s}{v}) \quad (17)$$

and taking the partial derivative of the first term, i.e.

$$\frac{\partial i}{\partial t} = \frac{e_1}{R_o} \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})}$$

Then

$$e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \left(-\frac{1}{v}\right) = -L \frac{e_1}{R_o} \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \quad (18)$$

$$\text{This is true if } R_o = \sqrt{\frac{L}{C}} \quad (19)$$

The $\sqrt{\frac{L}{C}}$ is called the characteristic resistance of the loss-less line and is designated as R_o or R_c .

Proceeding in a similar manner for ^{the} second terms of e and i results in $R_o = -\sqrt{\frac{L}{C}}$. However, in order to have a positive sign for R_o in both cases the solution for i becomes

$$i = \frac{e_1}{\sqrt{L/C}} f(t - \frac{s}{v}) - \frac{e_2}{\sqrt{L/C}} f(t + \frac{s}{v}) \quad (20)$$

Now inasmuch as $e_1 f(t - \frac{s}{v})$ is a wave travelling in the positive direction of s and $e_2 f(t + \frac{s}{v})$ is travelling in the negative s direction it is proper to call $e_1 f(t - \frac{s}{v})$ an incident wave and $e_2 f(t + \frac{s}{v})$ a reflected wave.

$$(1) \quad \frac{1}{x} = 1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \frac{1}{5}x^4 - \frac{1}{6}x^5 + \frac{1}{7}x^6 - \frac{1}{8}x^7 + \frac{1}{9}x^8 - \frac{1}{10}x^9 + \dots$$

3. Approximate value of $\ln 2$ by using the series (1) with $x = 1$.

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \dots$$

$$\frac{1}{x} = 1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \frac{1}{5}x^4 - \frac{1}{6}x^5 + \frac{1}{7}x^6 - \frac{1}{8}x^7 + \frac{1}{9}x^8 - \frac{1}{10}x^9 + \dots$$

$$\frac{1}{x} = 1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \frac{1}{5}x^4 - \frac{1}{6}x^5 + \frac{1}{7}x^6 - \frac{1}{8}x^7 + \frac{1}{9}x^8 - \frac{1}{10}x^9 + \dots$$

$$(2) \quad \frac{1}{x} = 1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \frac{1}{5}x^4 - \frac{1}{6}x^5 + \frac{1}{7}x^6 - \frac{1}{8}x^7 + \frac{1}{9}x^8 - \frac{1}{10}x^9 + \dots$$

4. Approximate value of $\ln 2$ by using the series (2) with $x = 1$.

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \dots$$

5. Approximate value of $\ln 2$ by using the series (3) with $x = 1$.

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \dots$$

$$(3) \quad \frac{1}{x} = 1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \frac{1}{5}x^4 - \frac{1}{6}x^5 + \frac{1}{7}x^6 - \frac{1}{8}x^7 + \frac{1}{9}x^8 - \frac{1}{10}x^9 + \dots$$

6. Approximate value of $\ln 2$ by using the series (4) with $x = 1$.

7. Approximate value of $\ln 2$ by using the series (5) with $x = 1$.

8. Approximate value of $\ln 2$ by using the series (6) with $x = 1$.

Hence

$$e = e_i + e_r \quad \text{where} \quad (21)$$

$$e_i = e_1 f(t - \frac{s}{v}) \quad \text{and} \quad e_r = e_2 f(t + \frac{s}{v})$$

Likewise

$$i = i_i - i_r \quad \text{where} \quad (22)$$

$$i_i = \frac{e_1}{R_o} f(t - \frac{s}{v}) \quad \text{and} \quad i_r = \frac{e_2}{R_o} f(t + \frac{s}{v})$$

$$\text{where } R_o = \sqrt{L/C}$$

Suppose now a loss-less line is terminated in a resistance R_L , then $e_L = i_L R_L$. Hence

$$\begin{aligned} e_L &= e_{iL} + e_{rL} \\ i_L &= \frac{e_{iL}}{R_o} - \frac{e_{rL}}{R_o} \end{aligned} \quad (23)$$

The solution of these two equations yields

$$\frac{e_{rL}}{e_{iL}} = \frac{R_L - R_o}{R_L + R_o} = K_{eL} \quad (24)$$

$$\frac{i_{rL}}{i_{iL}} = - \frac{R_L - R_o}{R_L + R_o} = K_{iL}$$

Equations 24 give the voltage reflection coefficient K_e and current reflection coefficient K_i both at ^{the} load resistance R_L . In other words

$$e_{rL} = K_{eL} e_{iL} \quad \text{and} \quad i_{rL} = K_{iL} i_{iL}$$

For the case in which $R_L = R_0$, $K_{eL} = K_{iL} = 0$ and there is no reflection. That is, all of the energy that reaches the load is dissipated in the load.

When the line is shorted at the load $R_L = 0$, $K_{eL} = -1$ and $K_{iL} = +1$. When the line is open at the load $R_L = \infty$, $K_{eL} = +1$ and $K_{iL} = -1$.

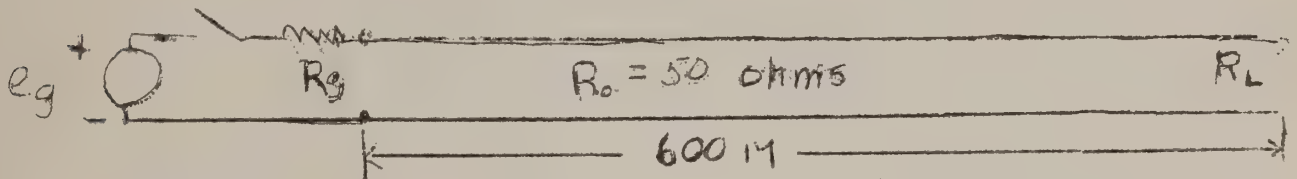


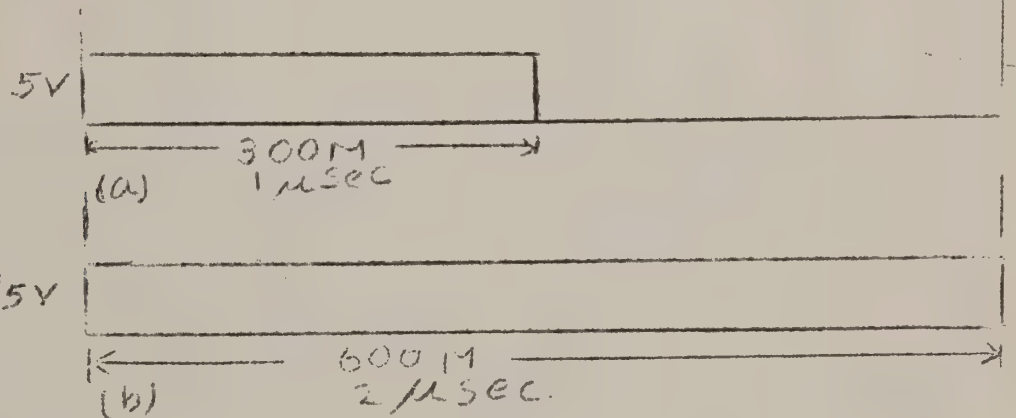
Fig. 6

Fig. 7

The source is a 10 volt battery.

$R_g = 50$ ohms

$R_L = 50$ ohms



Given a loss-less line, characteristic resistance $R_0 = 50$ ohms, 600 meters long illustrated by Fig. 6. The velocity of propagation is 300 meters per micro second or 3×10^8 meters per second.

Example 1. The source e_g is a 10 volt battery, the resistance $R_g = 50$ ohms and $R_L = 50$ ohms. Figure 7 illustrates the way in which the voltage propagates down the line. When the switch s is closed the resistance to the incident wave is 50 ohms, this is true regardless of the resistance of the load. Hence the line voltage is 5 volts. At the end of 1 μ second the voltage has propagated half way down the line. At the end of 2 μ seconds the entire line is raised to a 5 volt potential. The potential stays at 5 volts as long as the switch remains

closed. The current becomes $5/50 = .1$ ampere and propagates along with the voltage.

See page 13 for Fig. 8

Example 2. Suppose all conditions are same as in example 1 except $R_L = 150$ ohms. For this condition the voltage reflection coefficient $K_{eL} = \frac{150-50}{150+50} = +.5$ and the current reflection coefficient $K_{iL} = -.5$. Hence the incident voltage and reflected voltage at the load add up to 7.5 volts. Then at the end of 2 μ seconds +2.5 volts propagates toward the source. Since the source resistance $R_g = R_o$ there is no further reflections when the 2.5 volts reaches the source and the line remains at 7.5 volts as long as the switch is closed. The current will reach a steady value of $\frac{7.5}{150} = .05$ amperes. See Fig. 8.

See page 13 for Fig. 9

1. The first part of the report is a summary of the work done during the year.

2. The second part is a detailed account of the work done during the year.

3. The third part is a summary of the work done during the year.

4. The fourth part is a summary of the work done during the year.

5. The fifth part is a summary of the work done during the year.

6. The sixth part is a summary of the work done during the year.

7. The seventh part is a summary of the work done during the year.

8. The eighth part is a summary of the work done during the year.

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10. The tenth part is a summary of the work done during the year.

11. The eleventh part is a summary of the work done during the year.

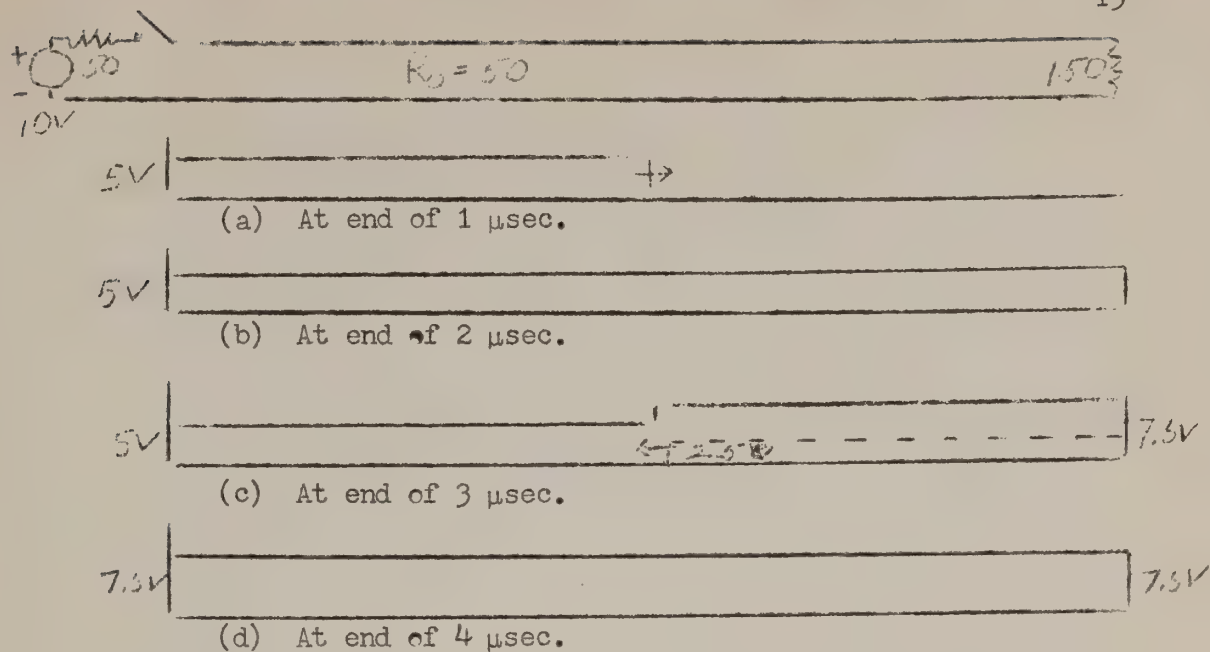


Fig. 8 $e_g = 10V$, $R_g = 50$ ohms, $R_L = 150$ ohms, for example 2
 $K_{eg} = 0$ and $K_{eL} = +0.5$

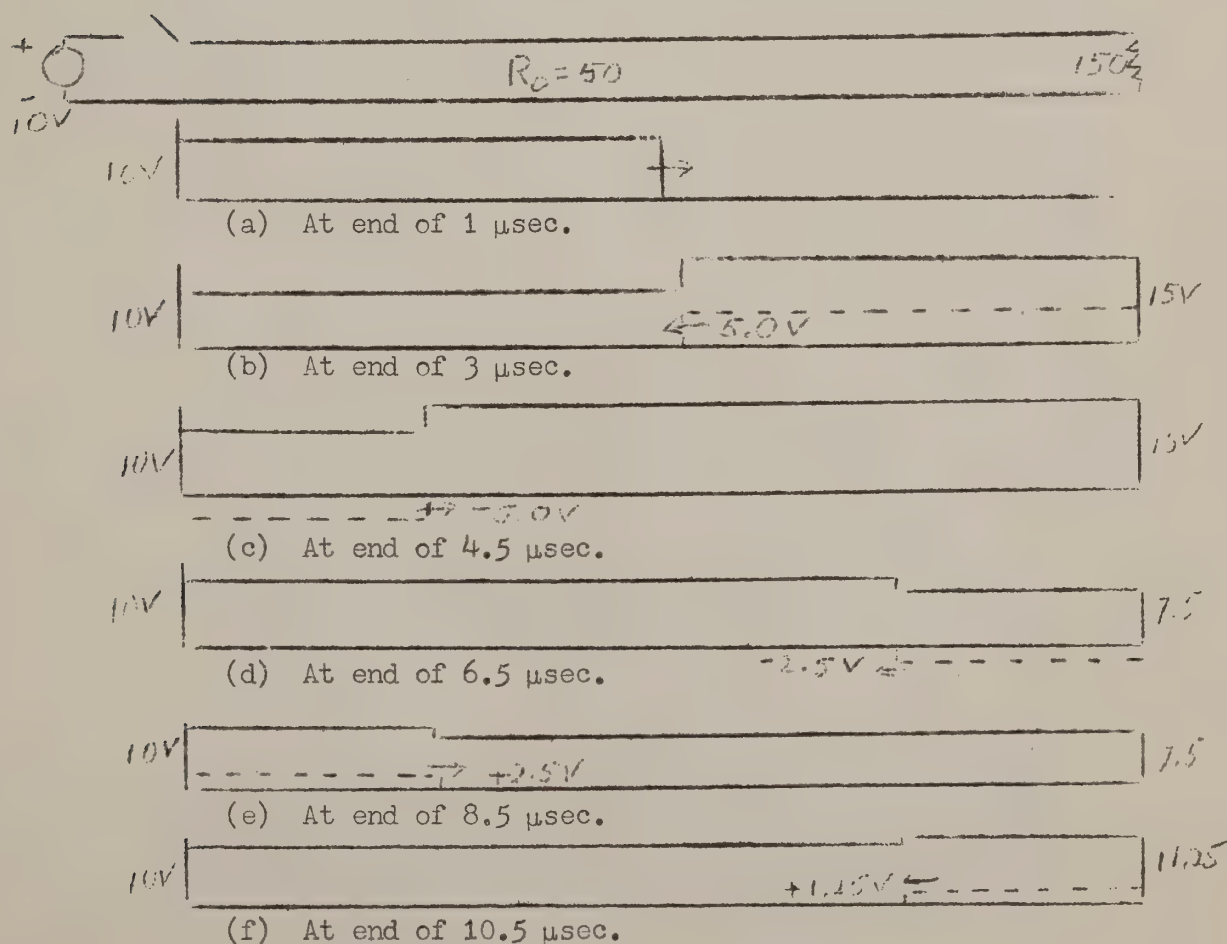


Fig. 9 $e_g = 10V$, $R_g = 0$, $R_L = 150$ ohms, for example 3
 $K_{eg} = -1$ and $K_{eL} = +0.5$

1. The first part of the paper is devoted to a general discussion of the problem of the existence of solutions of the system of equations

which is satisfied by the functions

where ϕ is a function of the variables x, y, z and t and ψ is a function of the variables x, y, z and t .

The second part of the paper is devoted to a detailed study of the case in which the functions ϕ and ψ are assumed to be of the form

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The third part of the paper is devoted to a study of the case in which the functions ϕ and ψ are assumed to be of the form

where ϕ and ψ are functions of the variables x, y, z and t and ϕ and ψ are functions of the variables x, y, z and t .

The fourth part of the paper is devoted to a study of the case in which the functions ϕ and ψ are assumed to be of the form

where ϕ and ψ are functions of the variables x, y, z and t and ϕ and ψ are functions of the variables x, y, z and t .

The fifth part of the paper is devoted to a study of the case in which the functions ϕ and ψ are assumed to be of the form

where ϕ and ψ are functions of the variables x, y, z and t and ϕ and ψ are functions of the variables x, y, z and t .

The sixth part of the paper is devoted to a study of the case in which the functions ϕ and ψ are assumed to be of the form

where ϕ and ψ are functions of the variables x, y, z and t and ϕ and ψ are functions of the variables x, y, z and t .

The seventh part of the paper is devoted to a study of the case in which the functions ϕ and ψ are assumed to be of the form

where ϕ and ψ are functions of the variables x, y, z and t and ϕ and ψ are functions of the variables x, y, z and t .

Example 3. Suppose the conditions are the same as in example 2 except $R_g = 0$. In this case the voltage reflection coefficient at the source becomes -1.0 . The load reflection coefficient remains at $+0.5$. Now events are pictured in Fig. 9. Graph (b) shows the first reflection of $+ .5 \times 10 = 5$ volts travelling toward the source. At the source -5 volts are reflected and travel toward the load. Graph (d) shows the second reflection of -2.5 volts travelling toward the source where $+2.5$ volts are reflected. The third reflection results in $+1.25$ volts travelling toward the load. Thus the reflections are dying out and the steady state voltage of the entire line becomes 10 volts.

Example 4. Suppose the conditions are the same as in example 1 except the 10 volt battery (step voltage) is replaced by a 20 volt pulse of $1 \mu\text{sec.}$ duration. Since $R_L = R_o$ this pulse which will reach the load in $2 \mu\text{seconds}$, will be dissipated entirely in the load resistor, no reflection.

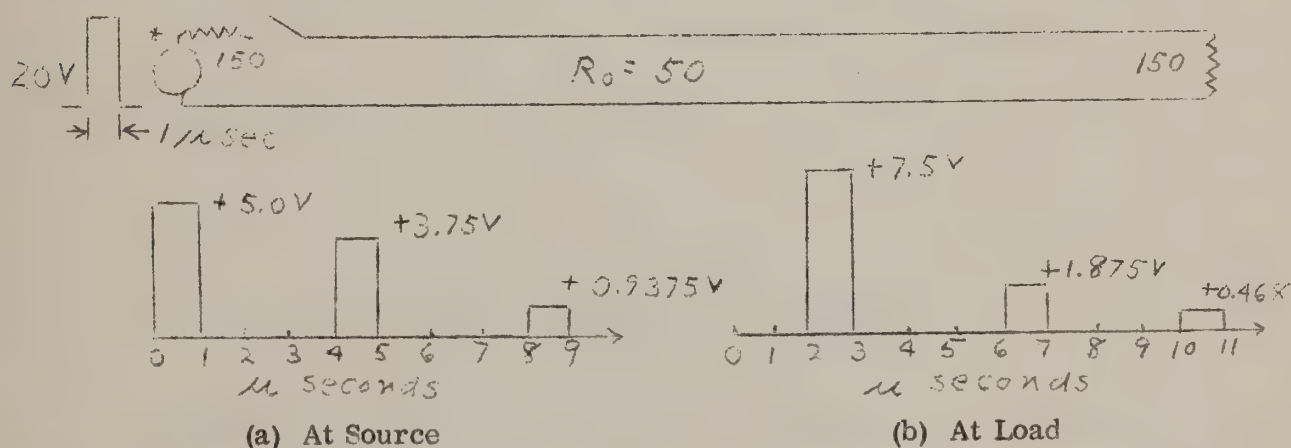


Fig. 10 For each reflection the incident and reflected voltages add to give the total voltage as shown. $K_{eg} = K_{eL} = +.5$ for example 5

Example 5. Suppose the conditions are the same as example 4 except the load resistance is 150 ohms and the source resistance is also 150 ohms. The voltage reflection coefficients at the load and at the source are both equal to 0.5. A $1 \mu\text{sec}$ pulse of 5 volts $[= 50 \times 20 / (150 + 50)]$ travels toward the load and reaches the load in $2 \mu\text{seconds}$. It is reflected at the load as a $+2.5$ volt pulse which travels toward and reaches the source end in another $2 \mu\text{seconds}$. It is reflected at the source as 1.25 volts travels toward the load and reaches the load in an

1. The first part of the paper is devoted to a general discussion of the problem of the origin of life. It is shown that the problem is one of the most important and most difficult in the history of science.

2. The second part of the paper is devoted to a detailed discussion of the various theories of the origin of life. It is shown that the most plausible theory is that of the origin of life from non-living matter.

3. The third part of the paper is devoted to a discussion of the evidence in support of the various theories of the origin of life. It is shown that the evidence is in favor of the theory of the origin of life from non-living matter.

4. The fourth part of the paper is devoted to a discussion of the implications of the various theories of the origin of life. It is shown that the theory of the origin of life from non-living matter has important implications for our understanding of the history of life on Earth.

5. The fifth part of the paper is devoted to a discussion of the future of research on the origin of life. It is shown that there is still much to be learned about the origin of life, and that the study of the origin of life is one of the most exciting and most important areas of research in the history of science.

6. The sixth part of the paper is devoted to a discussion of the philosophical implications of the various theories of the origin of life. It is shown that the theory of the origin of life from non-living matter has important philosophical implications for our understanding of the nature of life and the universe.

7. The seventh part of the paper is devoted to a discussion of the historical development of the various theories of the origin of life. It is shown that the study of the origin of life has a long and interesting history, and that the various theories of the origin of life have been developed over a period of many centuries.

additional 2 μ seconds and so on. Figure 10 represents the total voltages at the source and load for the first several μ seconds.

5. The Steady State Solution Using Phasor Notation.

Returning to the second order equations namely

$$\frac{\partial^2 e}{\partial s^2} = RGe + (RC + LG) \frac{\partial e}{\partial t} + LC \frac{\partial^2 e}{\partial t^2} \quad (5)$$

$$\frac{\partial^2 i}{\partial s^2} = RGi + (RC + LG) \frac{\partial i}{\partial t} + LC \frac{\partial^2 i}{\partial t^2} \quad (6)$$

Now replacing $\frac{\partial}{\partial t}$ by $j\omega$ ^{and $\frac{\partial^2}{\partial t^2}$} by $(j\omega)^2$ and writing V for e and I for i there results the equations in phasor form for the steady state when the voltage and current are varying sinusoidally with time namely

$$\frac{d^2 V}{ds^2} = RGV + (RC + LG) j\omega V - LC \omega^2 V \quad (25)$$

$$\frac{d^2 I}{ds^2} = RGI + (RC + LG) j\omega I - LC \omega^2 I \quad (26)$$

Now since

$$(R + j\omega L)(G + j\omega C) = ZY = RG - LC \omega^2 + j\omega (RC + LG)$$

$$\frac{\partial^2 V}{\partial s^2} = YZV \quad \text{or} \quad \frac{d^2 V}{ds^2} = YZV \quad (27)$$

$$\frac{\partial^2 I}{\partial s^2} = YZI \quad \text{or} \quad \frac{d^2 I}{ds^2} = YZI \quad (28)$$

Equations 27 and 28 are the second order differential equations for the lossy transmission line in phasor form. They apply only for the steady state when the voltages and currents are varying sinusoidally with time. The solution of 27 follows. It is assumed to be

$$V = V_1 e^{\gamma s} + V_2 e^{-\gamma s} \quad (29)$$

where V_1 , V_2 and γ are to be determined, the second derivative of 29 gives

$$\frac{\partial^2 V}{\partial s^2} = \gamma^2 V_1 e^{\gamma s} + \gamma^2 V_2 e^{-\gamma s} \quad (30)$$

Now according to equation 27 the right side of 30 must equal $ZYV = ZY(V_1 e^{\gamma s} + V_2 e^{-\gamma s})$ by virtue of 29. This will be so if $\gamma = \sqrt{ZY}$. Therefore the solution to 27 becomes

$$V = V_1 e^{\sqrt{ZY}s} + V_2 e^{-\sqrt{ZY}s} \quad (31)$$

where V_1 and V_2 are to be determined. Note that $\gamma = \sqrt{ZY}$ could have been obtained by substituting γ for $\frac{\partial}{\partial s}$ or γ^2 for $\frac{\partial^2}{\partial s^2}$ in equation 27.

Because $\cos(\omega t + \beta s)$ is involved in the solution of equation 5 in a non-phasor form. It is reasonable to expect if $j\omega$ can be substituted for $\frac{\partial}{\partial t}$ to get the phasor form that γ (which turns out to be the propagation constant) might also be substituted for $\frac{\partial}{\partial s}$. In other words $d^2V/ds^2 = \gamma^2 V$, similar to $\partial^2 V/\partial t^2 = (j\omega)^2 V$

A similar procedure for the current equation results in

$$I = I_1 e^{\sqrt{ZY}s} + I_2 e^{-\sqrt{ZY}s} \quad (32)$$

Now the voltages V_1 and V_2 and the currents I_1 and I_2 do not depend upon s but depend upon the terminal conditions. Suppose a reference point is chosen so that $I = I_0$ and $V = V_0$ at $s = 0$. s is positive in going from the reference point toward the load and negative in the opposite direction.

$$\text{Then } V = V_1 e^{\gamma \cdot 0} + V_2 e^{-\gamma \cdot 0} = V_1 + V_2 = V_0 \quad \text{at } s = 0 \quad (33)$$

and

using an equation that is the phasor form of 1 namely

$$\frac{dV}{ds} = -ZI \quad (34)$$

But from equation 31

$$\frac{dV}{ds} = V_1 \sqrt{ZY} e^{\sqrt{ZY}s} - V_2 \sqrt{ZY} e^{-\sqrt{ZY}s} \quad (35)$$

1. The first part of the paper is devoted to the study of the properties of the function $f(x)$ defined by the equation

where α is a real number, $\alpha > 0$. It is shown that the function $f(x)$ is continuous and differentiable at the point $x = 0$. The derivative of the function at this point is equal to α . The function $f(x)$ is also shown to be concave up for $x > 0$ and concave down for $x < 0$.

2. In the second part of the paper, the function $f(x)$ is studied for $x > 0$. It is shown that the function is increasing and concave up for $x > 0$.

3. The third part of the paper is devoted to the study of the function $f(x)$ for $x < 0$. It is shown that the function is decreasing and concave down for $x < 0$.

4. In the fourth part of the paper, the function $f(x)$ is studied for $x > 0$. It is shown that the function is increasing and concave up for $x > 0$.

$$f(x) = \frac{1}{\alpha} \ln(1 + \alpha x) \quad \text{for } x > 0$$

at $s = 0$ $\sinh \epsilon \sqrt{ZY} s = \epsilon^{-\sqrt{ZY} s} = 1$

$$V_1 \sqrt{ZY} - V_2 \sqrt{ZY} = -ZI_0 \quad (36)$$

$$V_1 - V_2 = -\sqrt{Z/Y} I_0 \quad (37)$$

Hence the solutions of 33 and 37 give

$$V_1 = 1/2 (V_0 - \sqrt{Z/Y} I_0)$$

$$V_2 = 1/2 (V_0 + \sqrt{Z/Y} I_0)$$

Consequently

$$V = 1/2 (V_0 + \sqrt{Z/Y} I_0) \epsilon^{-\gamma s} + 1/2 (V_0 - \sqrt{Z/Y} I_0) \epsilon^{+\gamma s} \quad (38)$$

In a similar manner

$$I = 1/2 (I_0 + \sqrt{Y/Z} V_0) \epsilon^{-\gamma s} + 1/2 (I_0 - \sqrt{Y/Z} V_0) \epsilon^{+\gamma s} \quad (39)$$

Equations 38 and 39 are the phasor equations for V and I at any point on a transmission line when the time variation is sinusoidal. V_0 and I_0 are the voltage and current at a reference point from which s is measured, positive toward the load end and negative toward the generator end. The unit for s is the same as that for the line parameters L, R, C and G .

6. Characteristic Impedance and Admittance

The quantities $\sqrt{Z/Y}$ and $\sqrt{Y/Z}$ appear in equations 38 and 39 for V and I . $\sqrt{Z/Y}$ is called the characteristic impedance and is designated as Z_0 .

Now since $Z = R + j\omega L$ and $Y = G + j\omega C$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = Z_0 = (\sqrt{Z/Y} (\cos \Psi + j \sin \Psi)) \quad (40)$$

$$= R_0 + jX_0 \text{ ohms, independent of } s$$

where

$$R_0 = \sqrt{Z/Y} \cos \Psi \quad (41)$$

$$X_0 = \sqrt{Z/Y} \sin \Psi$$

and

$$\Psi = \tan^{-1} \frac{\omega L}{R} - \tan^{-1} \frac{\omega C}{G} \quad (42)$$

1. The first part of the paper is devoted to a discussion of the

main results of the paper. The second part is devoted to a discussion of the

main results of the paper.

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main results of the paper. The fourth part is devoted to a discussion of the

main results of the paper. The fifth part is devoted to a discussion of the

main results of the paper.

Most transmission lines are so well insulated that (except for extremely low frequencies) $G \ll \omega C$. Also $\frac{\omega C}{G}$ is usually much larger than $\omega L/R$. Hence the angle of Z_0 lies between 0 and -45° for most all conditions. It is zero for the loss-less line and nearly -45° for small conductor cable type of line shown in Fig. 1d. For a pair of no. 10 copper conductors spaced 12 in. apart and operated 1000 cps $Z_0 = 692 \angle -11.75^\circ$. For this same line operated at 10^6 cps

$$Z_0 = 650 \angle \text{appx } 0^\circ$$

The angle is appx 0° because $\omega L/R = 230$ and $\omega C/G$ is assumed to be infinite. For a pair of number 19 conductors in a cable like Fig. 1d. and operated at 1000 cycles $Z_0 = 463 \angle -42^\circ$.

The characteristic admittance is symbolized by Y_0 and is the reciprocal of Z_0 .

7. The Propagation Constant γ .

The quantity γ appearing in the exponent of e , or $e^{\gamma s}$, is called the propagation constant and is equal to \sqrt{ZY} . Since, in general Z and Y are both complex, i.e. $Z = R + j\omega L$ and $Y = G + j\omega C$, γ will also be complex. It is composed of a real part α and a j , or imaginary part β , i.e.

$$\gamma = \alpha + j\beta = \sqrt{ZY} = \left| \sqrt{ZY} \right| \angle \phi \quad (43)$$

Then

$$\alpha = \left| \sqrt{ZY} \right| \cos \left(\frac{\tan^{-1} \frac{\omega L}{R} + \tan^{-1} \frac{\omega C}{G}}{2} \right) \quad \begin{array}{l} \text{attenuation constant} \\ \text{nepers/m} \end{array} \quad (44)$$

and

$$\beta = \left| \sqrt{ZY} \right| \sin \left(\frac{\tan^{-1} \frac{\omega L}{R} + \tan^{-1} \frac{\omega C}{G}}{2} \right) \quad \begin{array}{l} \text{phase constant} \\ \text{Radians/m} \end{array} \quad (45)$$

note the similarity between angle of Z_0 and the angle of γ

In other forms it may be shown that

$$\alpha = \sqrt{1/2 (|ZY| + GR - \omega^2 LC)} \quad (46)$$

$$\beta = \sqrt{1/2 (|ZY| - GR + \omega^2 LC)} \quad (47)$$

where $|ZY| = \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}$

Now since $\gamma = \alpha + j\beta$ the quantities $e^{\gamma s}$ and $e^{-\gamma s}$ appearing in the equations for V and I become $e^{\alpha s} e^{j\beta s}$ and $e^{-\alpha s} e^{-j\beta s}$ respectively. Thus equations 38 and 39 may be written as

$$V = 1/2(V_0 + Z_0 I_0) e^{-\alpha s} e^{-j\beta s} + 1/2(V_0 - Z_0 I_0) e^{\alpha s} e^{j\beta s} \quad (48)$$

$$I = 1/2(I_0 + Y_0 V_0) e^{-\alpha s} e^{-j\beta s} + 1/2(I_0 - Y_0 V_0) e^{\alpha s} e^{j\beta s} \quad (49)$$

At any point on the line V and I are varying sinusoidally (or consinusoidally) with time.

When a voltage or current is expressed in phasor form the consinusoidally time varying form may be obtained by taking the real part of ^{the} expression for V, or I, multiplied by $e^{j\omega t}$. Without going into much detail, this will result in

$$e = e_1 e^{-\alpha s} \cos(\omega t - \beta s + \delta i) + e_2 e^{\alpha s} \cos(\omega t + s + \delta r)$$

for the voltage, where e_1 and e_2 are obtained by evaluating

$$\text{Real part of } \frac{\sqrt{2}}{2} (V_0 + I_0 Z_0) e^{j(\omega t - \beta s + \delta i)}$$

and the real part of

$$\frac{\sqrt{2}}{2} (V_0 - I_0 Z_0) e^{j(\omega t + \beta s + \delta r)}$$

because

$$e^{j(\omega t \pm \beta s)} = \cos(\omega t \pm \beta s) + j \sin(\omega t \pm \beta s)$$

8. Wave Characteristics of Propagation on a Transmission Line

Equations 48 and 49 are the phasor equations for the steady state voltage and current of a general transmission ^{line} when driven by a sinusoidal e.m.f. Each equation is made up of two terms that have physical significance. The terms that have exponential factors with negative exponents are called incident components and those with positive exponents are called reflected components. Incident components constitute wave components that travel from source toward the load and reflected components travel in the opposite direction. Reflected components originate at the load or any discontinuity at which V/I is not equal to Z_0 . Thus

$$\begin{aligned}
 \text{an incident component of voltage} &= \frac{1}{2} (V_o + I_o Z_o) e^{-\alpha s} e^{-j\beta s} \\
 \text{a reflected component of voltage} &= \frac{1}{2} (V_o - I_o Z_o) e^{\alpha s} e^{j\beta s} \\
 \text{an incident component of current} &= \frac{1}{2} (I_o + V_o Y_o) e^{-\alpha s} e^{-j\beta s} \\
 \text{a reflected component of current} &= \frac{1}{2} (I_o - V_o Y_o) e^{\alpha s} e^{j\beta s}
 \end{aligned}$$

Incident components are largest at the source end and decrease exponentially as the point of interest moves toward the load end. They also shift in phase in a negative, or clockwise, direction. For example, Fig. 11(a) is a phasor diagram for incident voltages at intervals of 30° ($1/12$ wavelength) for a complete 360° rotation or one wavelength of line. The line is open at the load end.

Reflected components of V and I are largest at the load and decrease in magnitude as the point of interest moves toward the generator. They shift in phase in a positive direction in going from the generator to the load.

Figure 11(b) is a phasor diagram for the reflected components of voltage on a transmission line open at the load end. The total phasor voltage is the phasor sum of the incident reflected voltage at any point on the line.

Now instantaneous voltage may be derived from a phasor voltage diagram by rotating the phasor diagram uniformly counter-clockwise with time and plotting the projection of the phasor on the horizontal axis as ordinates and angle of rotation as abscissae. This gives a cosine function of ωt . At time $t = 0$, when V is a maximum at the generator the projection of each of the phasors shown in Fig. 11(a) on the horizontal axis represent the voltage existing at locations βs degrees down the line. For example, phasor 9 is located $\beta s = 90^\circ$ from the generator and the instantaneous voltage is zero when the generator voltage, phasor 12, is a maximum. The voltage at location 9 reaches its maximum 90 time degrees later. However it is smaller in magnitude because of energy loss due to R and G on the line between the generator and position 9.

Figure 12 represents the instantaneous voltages, incident and reflected, on a transmission line 2 wave lengths long, open at the load end and moderate attenuation. The time interval between the different graphs of this figure is $1/8$ of a period. Note that the incident voltage propagates toward the load and the reflected voltage toward the generator. The 5 graphs covers a time interval of $1/2$ period. The total voltage at any time and at any point on the

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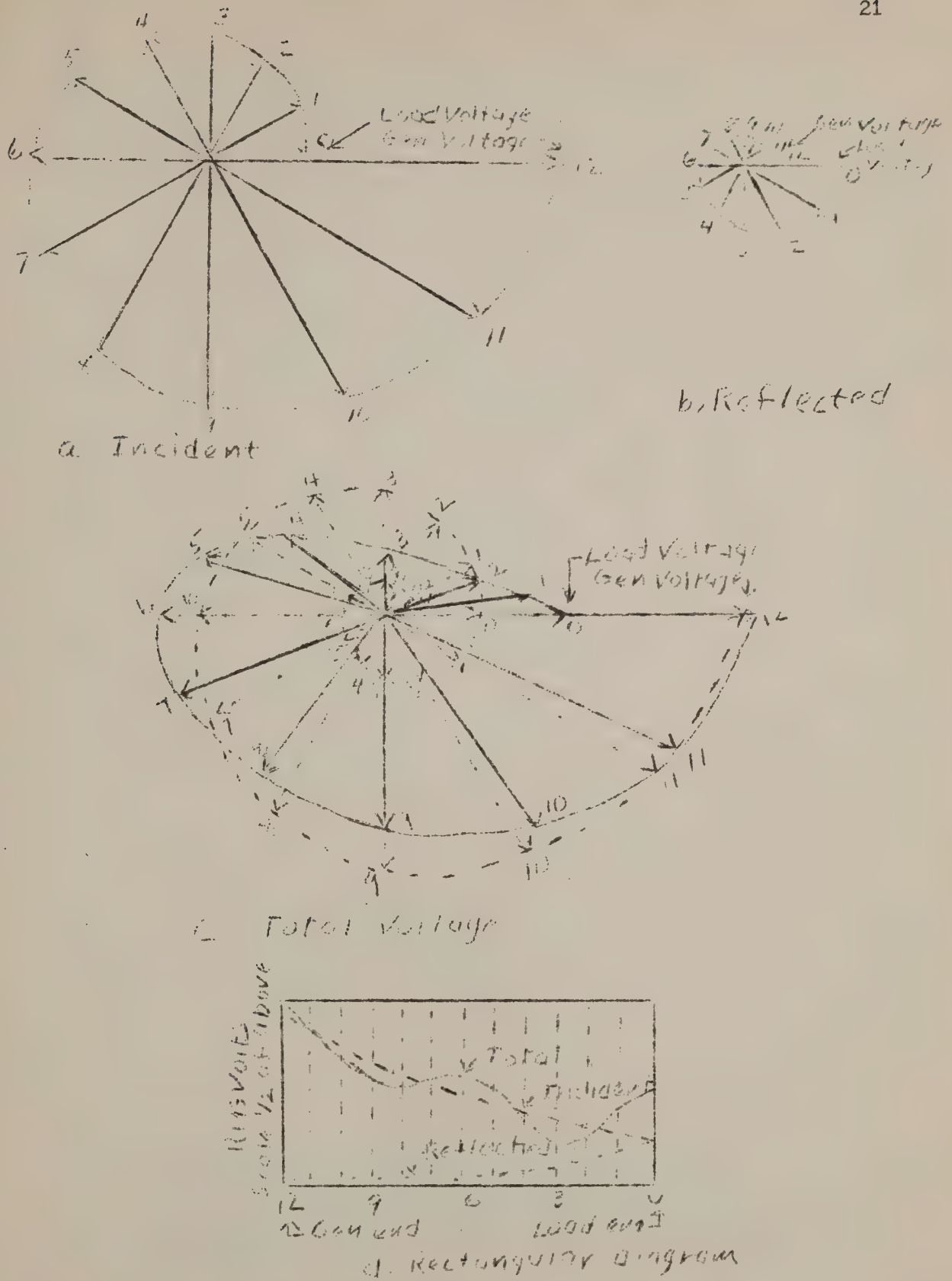


Fig. 11. Diagram, Phasor and rectangular, of incident and reflected components of voltage on a transmission line with load end open. The interval between phasors is $\beta s = 30^\circ$ or $1/12$ of a wave length.

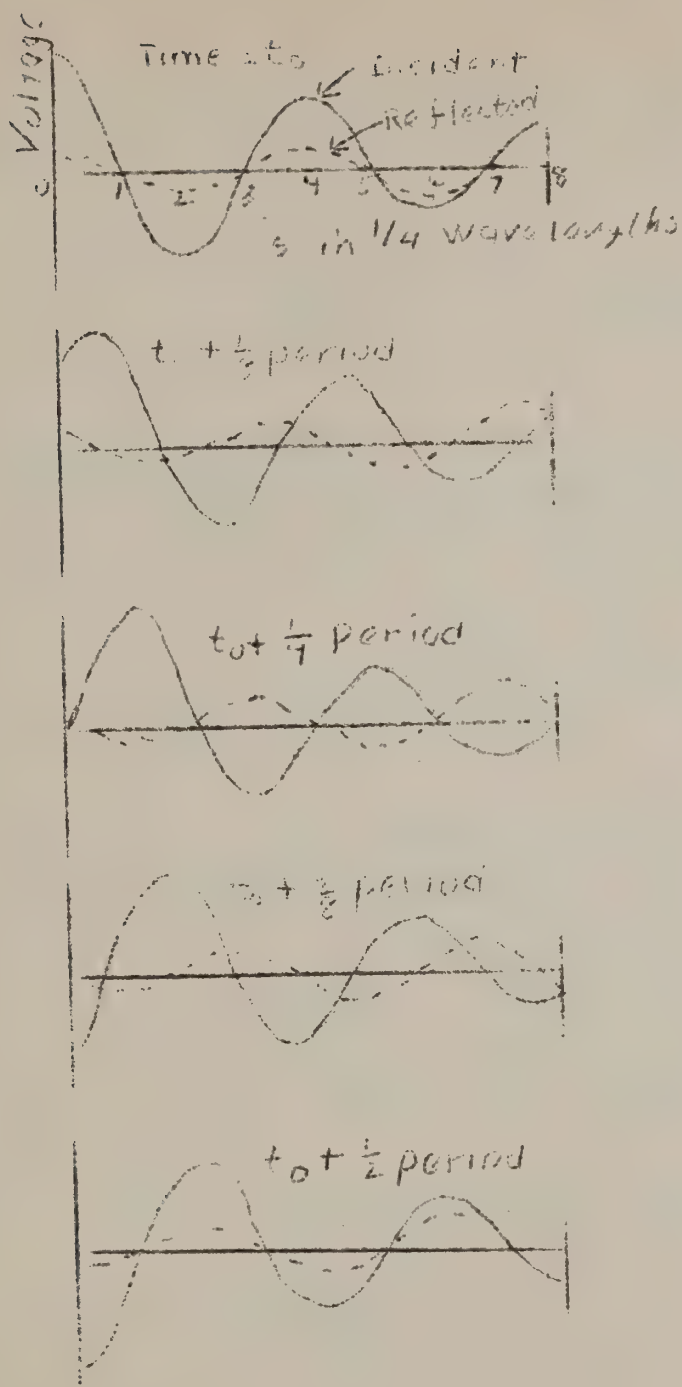


Fig. 12. Distribution of incident and reflected waves of instantaneous voltages on a lossy transmission line for various instants of time. Solid curves are incident voltages. Dotted Curves are reflected voltages.

line is the algebraic sum of the incident and reflected voltages. This voltage is given by

$$e = \text{real part of } \frac{\sqrt{2}}{2} e^{-\alpha s} (V_o + I_o Z_o) e^{j(\omega t - \beta s)} \\ + \text{real part of } \frac{\sqrt{2}}{2} e^{\alpha s} (V_o - I_o Z_o) e^{j(\omega t + \beta s)}$$

9. Input Impedance of a Lossy Line

Referring to equations 48 and 49 and measuring distance from the load end the input impedance toward the load at any distance s from the load i.e. the ratio of V to I becomes

$$Z_i = Z_o \frac{(Z_L + Z_o) + (Z_L - Z_o) e^{-2\alpha s} / -2\beta s}{(Z_L + Z_o) - (Z_L - Z_o) e^{-2\alpha s} / -2\beta s} \quad (50)$$

where s is the distance from the load to the point in question, and its negative aspect is already incorporated in the equation. It is seen that when

$(Z_L - Z_o) e^{-2\alpha s} / -2\beta s$ is zero or very small compared to $(Z_L + Z_o)$ then $Z_i = Z_o$. This is true when $Z_L = Z_o$ and also when $e^{-2\alpha s}$ is very small because of a comparative high αs product. When $Z_L = Z_o$ there is no reflected wave. When αs is large the energy reaching the load is so small that the reflected wave has very little effect upon the input impedance.

When the load end is short circuited, i.e., $Z_L = 0$

$$Z_{is} = Z_o \frac{1 - e^{-2\alpha s} / -2\beta s}{1 + e^{-2\alpha s} / -2\beta s} \quad (51) \\ = Z_o \tanh \gamma s$$

Examination of the absolute value of this equation shows that $Z_i > Z_o$ when $\beta s = \pi/2$ and its odd multiples and $Z_i < Z_o$ when $\beta s = \pi$ and its integral multiples including 0 when α is very small Z_i becomes very large at odd multiples of $\beta s = \frac{\pi}{2}$ and very small at integral multiples of $\beta s = \pi$.

When the line is open at the load, i.e., $Z_L = \infty$

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$$Z_{io} = Z_o \frac{1 + e^{-2\alpha s} \underline{-2\beta s}}{1 - e^{-2\alpha s} \underline{-2\beta s}} \quad (52)$$

$$= Z_o \coth \gamma s$$

Thus where Z_{is} is large Z_{io} is small and vice-versa.

Now from four terminal network theory

$\sqrt{Z_{io}Z_{is}}$ = image impedance of the network which is the same as the characteristic impedance for a symmetrical T or π network.

Likewise

$$\sqrt{Z_{io}Z_{is}} = \sqrt{Z_o^2} = Z_o \quad \text{for a transmission line.} \quad (53)$$

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HIGH FREQUENCY TRANSMISSION LINES

1. Line Parameters at High Frequencies

When transmission lines are used at high frequencies generally the construction and configuration are such that analysis is simplified because certain approximations may be made without much error. Due to skin effect of the current in a conductor it may be shown that

$$R_{ow} = \frac{1}{r} \sqrt{\frac{f\mu}{\pi\sigma}} \text{ ohms per meter length of line for an open wire line (For both wires)} \quad (1)$$

$$R_c = \left(\frac{1}{r_i} + \frac{1}{r_o} \right) \sqrt{\frac{f\mu}{4\pi\sigma}} \text{ ohms per meter length of line for a coaxial line} \quad (2)$$

where r is radius of the conductor in meters, r_i is the radius of the inner conductor in meters and r_o is the inner radius of the outside conductor in meters. σ is the conductivity in mhos/meter and μ is the permeability. $\mu = 4\pi \times 10^{-7}$ for non-magnetic conductors, σ is the conductivity in mhos/meter and μ is the permeability. $\mu = 4\pi \times 10^{-7}$ for non-magnetic conductors, $\sigma = 5.8 \times 10^7$ for copper. f is in cycles per second (cps).

Now, inasmuch as R is proportional to \sqrt{f} and ωL is proportional to f at high frequencies, $\omega L \gg R$ with the result that for much of the analysis except where energy loss is involved R may be neglected. Generally, lines are so well insulated that G is negligible compared to ωC .

Also due to skin effect the magnetic flux linkages inside the conductors are negligible and the inductance and capacitance are given by

$$L_{ow} = 9.21 \times 10^{-7} \log \frac{D}{r} \text{ henries per meter open wire} \quad (3)$$

$$L_c = 4.6 \times 10^{-7} \log \frac{r_o}{r_i} \text{ henries per meter coaxial} \quad (4)$$

$$C_{ow} = \frac{12.05 \epsilon_r}{\log \frac{D}{r}} \mu\mu f \text{ per meter open wire} \quad (5)$$

$$C_c = \frac{24.1 \epsilon_r}{\log \frac{r_o}{r_i}} \text{ coaxial} \quad (6)$$

Note for either type of line

$$\frac{1}{\sqrt{LC}} = c / \sqrt{\epsilon_r} = v_p \quad (7)$$

THEORY OF THE EARTH

The theory of the earth is a branch of geology which deals with the origin and development of the earth and its various parts. It is a science which seeks to explain the causes of the various geological phenomena which we observe in nature.

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where ϵ_r is the dielectric constant of the material surrounding the line. For air $\epsilon_r = 1$. Also

$$Z_o = \sqrt{L/C} \quad (8)$$

$$Z_o = 276 \log \frac{D}{r} \text{ ohms for open wire lines} \quad (9)$$

$$Z_o = \frac{138}{\sqrt{\epsilon_r}} \log \frac{r_o}{r_i} \text{ ohms for coaxial line} \quad (10)$$

$$Z_o = \frac{276}{\sqrt{\epsilon_r}} \log \left(\frac{D}{r} \frac{1 - (\frac{D}{d})^2}{1 + (\frac{D}{d})^2} \right) \text{ for a balanced shielded line} \quad (11)$$

where D is the distance between conductors and d is the inside diameter of the metallic shield, both in meters.

The wave length λ is

$$\lambda = \frac{3 \cdot 10^8}{f \sqrt{\epsilon_r}} \quad (12)$$

The propagation constant is

$$\gamma = j\omega \sqrt{LC} = j\beta = j \frac{\omega}{v_p} \quad (13)$$

For an analysis that involves the losses or the efficiency of transmission α cannot be set equal to 0 and β is not exactly equal to $\omega \sqrt{LC}$. Now since $R \ll \omega L$ and $G = \omega DC \ll \omega C$, where D is the dissipation factor of the dielectric which for low loss dielectrics is approximately equal to the power factor

$$\alpha = \sqrt{\frac{1}{2} (|Z Y| + GR - BX)} \quad (14)$$

$$= \sqrt{\frac{1}{2} \left[(\omega^2 C^2 + G^2)^{1/2} (\omega^2 L^2 + R^2)^{1/2} + GR - BX \right]} \quad (15)$$

Expanding $(\omega^2 C^2 + G^2)^{1/2}$ and $(\omega^2 L^2 + R^2)^{1/2}$ and neglecting all terms of each beyond the second, there results

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$$\alpha = \sqrt{\frac{1}{2} \left[\left(\omega C + \frac{G^2}{2\omega C} \right) \left(\omega L + \frac{R^2}{2\omega L} \right) + GR - B^2 \right]} \quad (16)$$

$$\alpha = \sqrt{\frac{1}{2} \left[\left(\omega^2 LC + \frac{R^2 G^2}{2\omega^2 LC} + \frac{R^2 C}{2L} + \frac{G^2 L}{2C} \right) + GR - \omega^2 LC \right]} \quad (17)$$

Neglecting $\frac{R^2 G^2}{2\omega^2 LC}$ in comparison to GR

$$\alpha \approx \sqrt{\frac{1}{2} \left(\frac{R^2 C}{2L} + \frac{G^2 L}{2C} + GR \right)} \quad (18)$$

$$\approx \frac{R}{2} \sqrt{\frac{C}{L}} + \frac{G}{2} \sqrt{\frac{L}{C}} = \frac{R}{2Z_0} + \frac{GZ_0}{2} \quad (19)$$

where $Z_0 = \sqrt{L/C}$ is the characteristic impedance of the loss-less line.

In a similar manner

$$\beta \approx \omega \sqrt{LC} \left(1 + \frac{\alpha^2}{2\omega^2 LC} \right) \quad (20)$$

$$\approx \beta_0 \left(1 + \frac{\alpha^2}{2\beta_0^2} \right) \quad (21)$$

where β_0 is the strictly lossless phase constant. Consequently the phase velocity (ω/β) is less than velocity of light. For both open wire and coaxial lines some loss occurs in the insulators. This is difficult to evaluate analytically. Manufacturers of coaxial lines usually supply the total attenuation for lines generally in decibels per 100 feet for the larger size lines.

2. Propagation on a Loss-less Line

High frequency transmission lines are usually very short physically but may be several wave lengths long electrically. For a strictly lossless line $\alpha = 0$, $\gamma = j\beta = j\omega \sqrt{LC}$, $Z_0 = \sqrt{L/C}$, $\lambda_0 = v_p/f$ and $v_p = \omega/\beta = c/\sqrt{\epsilon_r}$. Then referring to equation 48 of Transmission Line Theory

$$V = \frac{1}{2} (V_0 + Z_0 I_0) e^{-j\beta s} + \frac{1}{2} (V_0 - Z_0 I_0) e^{+j\beta s}$$

Now put $V_0 = V_L$, the load voltage, and $I_0 = I_L = V_L/Z_L$ where Z_L is the load

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impedance and measure distance from the load

$$V = \frac{1}{2} V_L (1 + Z_o/Z_L) e^{j\beta s} + \frac{1}{2} V_L (1 - Z_o/Z_L) e^{-j\beta s} \quad (22)$$

which can be written as

$$V = \frac{V_L (Z_L + Z_o)}{2Z_L} (e^{j\beta s} + \frac{Z_L - Z_o}{Z_L + Z_o} e^{-j\beta s}) \quad (23)$$

$\frac{Z_L - Z_o}{Z_L + Z_o}$ is the load end voltage reflection coefficient K_L , i.e., K_L is ratio of reflected voltage V_r to the incident voltage V_i at the load. Hence

$$V = \frac{V_L (Z_L + Z_o)}{2Z_L} (e^{j\beta s} + K_L e^{-j\beta s}) \quad (24)$$

where s is measured from the load end and numerical values are used in the equation because the signs have already been changed.

In a similar manner

$$I = \frac{V_L (Z_L + Z_o)}{2Z_L Z_o} (e^{j\beta s} - K_L e^{-j\beta s}) \quad (25)$$

Note K_L is the voltage reflection coefficient and $(-K_L)$ is the current reflection coefficient. Now equations 24 and 25 are in the form of two travelling waves, one moving from generator to load and the other moving from load to generator, i.e., $e^{j\beta s}$ and $e^{-j\beta s}$ respectively.

Since $e^{\pm j\beta s} = \cos \beta s \pm j \sin \beta s$, equation 24 and likewise 25, may be put into two other forms, namely

$$V = \frac{V_L (Z_L + Z_o)}{2Z_L} [(1 - K_L) e^{j\beta s} + 2K_L \cos \beta s] \quad (26)$$

and

$$V = \frac{V_L (Z_L + Z_o)}{2Z_L} [(1 + K_L) \cos \beta s + j (1 - K_L) \sin \beta s] \quad (27)$$

Equation 26 indicates a travelling wave component associated with $e^{j\beta s}$ and a standing wave component associated with $\cos \beta s$. Equation 27 indicates two standing wave components out of space phase and out of time phase.

1.

1. The first part of the paper is devoted to a general discussion of the problem.

2.

2. In the second part, we consider the case of a single particle.

3.

3. The third part is devoted to the case of a system of particles.

4.

4. In the fourth part, we consider the case of a system of particles.

5.

5. The fifth part is devoted to the case of a system of particles.

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6. The sixth part is devoted to the case of a system of particles.

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7. The seventh part is devoted to the case of a system of particles.

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8. The eighth part is devoted to the case of a system of particles.

9.

9. The ninth part is devoted to the case of a system of particles.

When $K_L = 0$, i.e., $Z_L = Z_0$, equations 24 and 26 are of the same form because there is only one travelling wave moving from generator to load. Equation 27 indicates two standing waves 90° out of space phase and 90° out of time phase. This is equivalent to the form of equation 24 which shows that the absolute magnitude of V is V_L at all points on the line and undergoes only sinusoidal variations in time. When $K_L = 1$, i.e., $Z_L = \infty$, equation 27 indicates a pure standing wave where V is a cosine function of s along the line. A similar situation exists when $K_L = -1$, i.e., $Z_L = 0$, where V is a sine function of s .

When $Z_L = Z_0$ the equations 24 and 25 for V and I show that V and I are in phase everywhere along the line including the load. Also $V/I = Z_0$. When $K_L = 1$ or -1 the voltage and current are 90° out of both space and time phase. In this case no energy is transmitted and there is no further wave motion when the steady state is reached. This is also true when Z_L is a pure reactance. Then the absolute value of K_L is 1 but there is a phase angle.

Now returning to equation 26, namely

$$V = \frac{V_L(Z_L + Z_0)}{2Z_L} [(1 - K_L) e^{j\beta s} + 2K_L \cos \beta s]$$

and plotting the two terms of this equation for various instants of time the curves of Fig. 1 result. Here $K_L = .5 \angle 0$ and time is reckoned from time that both the travelling and standing wave components are a positive maximum at the generator end of the line. At points, a, c, e and g the magnitude of $[(1 - K_L) e^{j\beta s} + 2K_L \cos \beta s]$ is 1.5 and it varies cosinusoidally with time. At points b, d and f the magnitude is .5. Thus a standing wave pattern is created with a maximum value of 1.5 and a minimum value of .5.

The ratio of V_{\max} and V_{\min} is called the standing ratio S .

Now returning to equation 24 and writing $e^{j\phi_L}$ for the angle of K_L and $e^{j\theta}$ for the angle of $V_L(Z_L + Z_0)/2Z_L$ there results

$$V = \left| \frac{V_L(Z_L + Z_0)}{2Z_L} \right| e^{j(\theta + \beta s)} (1 + |K_L| e^{j(\phi_L - 2\beta s)}) \quad (29)$$

Now the magnitude of $\left| \frac{V_L(Z_L + Z_0)}{2Z_L} \right| e^{j(\theta + \beta s)}$ does not change with s . Hence any amplitude change in V with s is contained in $(1 + |K_L| e^{j(\phi_L - 2\beta s)})$ which has the absolute value $\sqrt{1 + |K_L|^2 + 2|K_L| \cos(\phi_L - 2\beta s)}$.

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9. The ninth part is devoted to a detailed analysis of the case of a system of particles.

10. The tenth part is devoted to a detailed analysis of the case of a system of particles.

11. The eleventh part is devoted to a detailed analysis of the case of a system of particles.

12. The twelfth part is devoted to a detailed analysis of the case of a system of particles.

13. The thirteenth part is devoted to a detailed analysis of the case of a system of particles.

14. The fourteenth part is devoted to a detailed analysis of the case of a system of particles.

15. The fifteenth part is devoted to a detailed analysis of the case of a system of particles.

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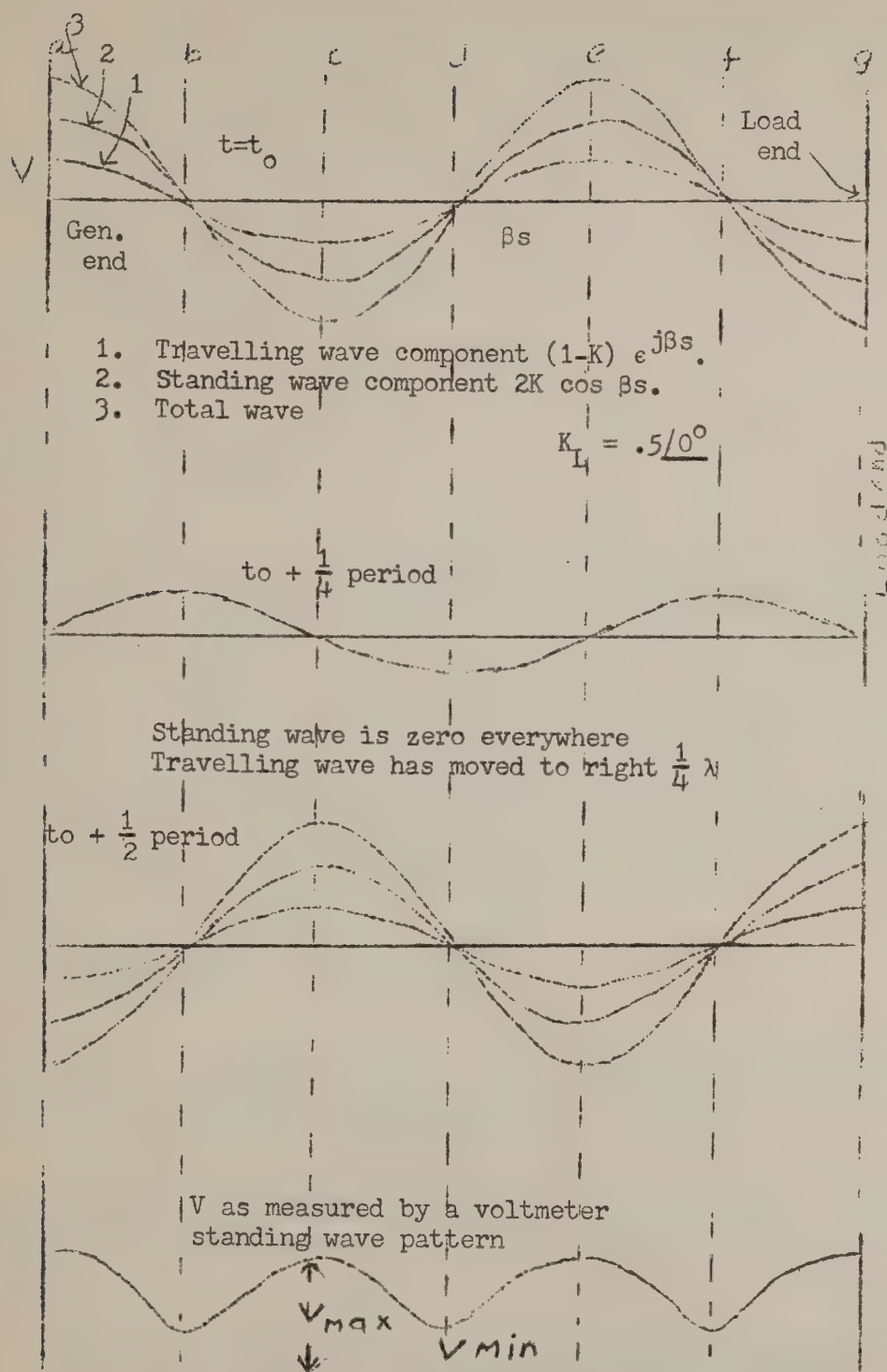


Fig. 1 Illustrating how a standing wave pattern of voltage on a transmission line is created by a pure travelling wave component plus a pure standing wave component.

$$V = \frac{V_L (Z_L + Z_0)}{2Z_L} [(1-K) e^{j\beta s} + 2K_L \cos \beta s]$$

Now $\sqrt{1 + |K_L|^2 + 2|K_L| \cos(\phi_L - 2\beta s)}$ is a maximum when $\phi_L - 2\beta s = 2n\pi$ and a minimum when $\phi_L - 2\beta s = (2n-1)\pi$ where n is an integer including 0. Hence

$$|V_{\max}| = \left| \frac{V_L(Z_L + Z_0)}{2Z_L} \right| (1 + |K_L|) \quad (30)$$

$$|V_{\min}| = \left| \frac{V_L(Z_L + Z_0)}{2Z_L} \right| (1 - |K_L|) \quad (31)$$

Then

$$\frac{|V_{\max}|}{|V_{\min}|} = \frac{1 + |K_L|}{1 - |K_L|} = S \quad (32)$$

where S is called the standing wave ratio and is a measure of the mismatch between the load impedance Z_L and the characteristic impedance Z_0 . Values for S range from 1 to ∞ . $S = 1$ when $|K_L| = 0$ and $|K_L| = 0$ when $Z_L = Z_0$. $S = \infty$ when $|K_L| = 1$, and $|K_L| = 1$ when $Z_L = 0$ or ∞ or is a pure reactance of any size. In other words S is infinite when the load impedance is such that no power is dissipated. For the graphs of Fig. 1, since $|K_L| = .5$, $S = \frac{1+.5}{1-.5} = 3$.

Because of the relation between $|K_L|$ and S as indicated by equation 32 if one is known the other may be determined. For example,

$$S = \frac{1 + |K_L|}{1 - |K_L|} \quad \text{or} \quad |K_L| = \frac{S-1}{S+1} \quad (33)$$

However this does not give the angle of K_L . The angle of K_L may be obtained from further measurements as explained in a later section.

3. Input Impedance of a Loss-less Line

The input impedance toward the load at any distance s from the load is equal to V_i/I_i and is given by

$$Z_i = Z_0 \frac{e^{j\beta s} + K_L e^{-j\beta s}}{e^{j\beta s} - K_L e^{-j\beta s}} = Z_0 \frac{Z_L \cos \beta s + j Z_0 \sin \beta s}{Z_0 \cos \beta s + j Z_L \sin \beta s} \quad (34)$$

which may be written as

$$Z_i = R_i + j X_i = Z_0 \frac{1 - |K_L|^2 + j 2|K_L| \sin(\phi_L - 2\beta s)}{1 + |K_L|^2 - 2|K_L| \cos(\phi_L - 2\beta s)} \quad (35)$$

• • •

where

$$R_i = Z_o \frac{1 - |K_L|^2}{1 + |K_L|^2 - 2|K_L| \cos(\phi_L - 2\beta s)} \quad (36)$$

$$X_i = Z_o \frac{+ 2|K_L| \sin(\phi_L - 2\beta s)}{1 + |K_L|^2 - 2|K_L| \cos(\phi_L - 2\beta s)} \quad (37)$$

Now $\cos(\phi_L - 2\beta s)$ ranges between +1 and -1. Hence R_i ranges between $Z_o \frac{1 - |K_L|}{1 + |K_L|}$ and $Z_o \frac{1 + |K_L|}{1 - |K_L|}$ which is $\frac{Z_o}{S}$ and $Z_o S$ respectively.

$$R_i = Z_o \frac{1 + |K_L|}{1 - |K_L|} = Z_o S \quad \text{for } (\phi_L - 2\beta s) = 2n\pi \quad (38)$$

and

$$R_i = \frac{Z_o}{S} \quad \text{for } \phi_L - 2\beta s = (2n-1)\pi \quad (39)$$

For these values of $(\phi_L - 2\beta s)$, $\sin(\phi_L - 2\beta s)$ is zero and $Z_i = R_i$, i.e., $X_i = 0$. When $(\phi_L - 2\beta s)$ is such as to make $\sin(\phi_L - 2\beta s)$ positive then X_i is an inductive reactance, and a capacitive reactance when $\sin(\phi_L - 2\beta s)$ is negative. A more detailed analysis will show that where the standing wave pattern slopes down toward the load end the reactance is inductive and where it slopes up the reactance is capacitive. The magnitude of the reactance lies between 0 and $Z_o (2|K_L|/1 - |K_L|^2)$. This is not usually important.

Figure 2 illustrates some of the pertinent information regarding input resistances and reactances. When a loss-less line is terminated in Z_o the input impedance at all points is equal to Z_o . When the line is open, shorted or terminated in a reactance $S = \infty$ and Z_i is a pure reactance everywhere.

It is interesting to note that if a resistance equal to $Z_o S$ were connected across the line at any of the points a, c, e and g and the rest of line toward the load were disconnected the standing wave pattern toward the generator would remain unchanged. A similar situation, of course, is true for connecting a resistance Z_o/S across the line at points b, d and f.

4. Determination of the Load Impedance Z_L from Observations on a Loss-less Line

Referring back to the expression for the absolute value of V on a loss-less line terminated in Z_L it is seen that minimums of V occur when

$$\phi_L - 2\beta s = (2n-1)\pi$$

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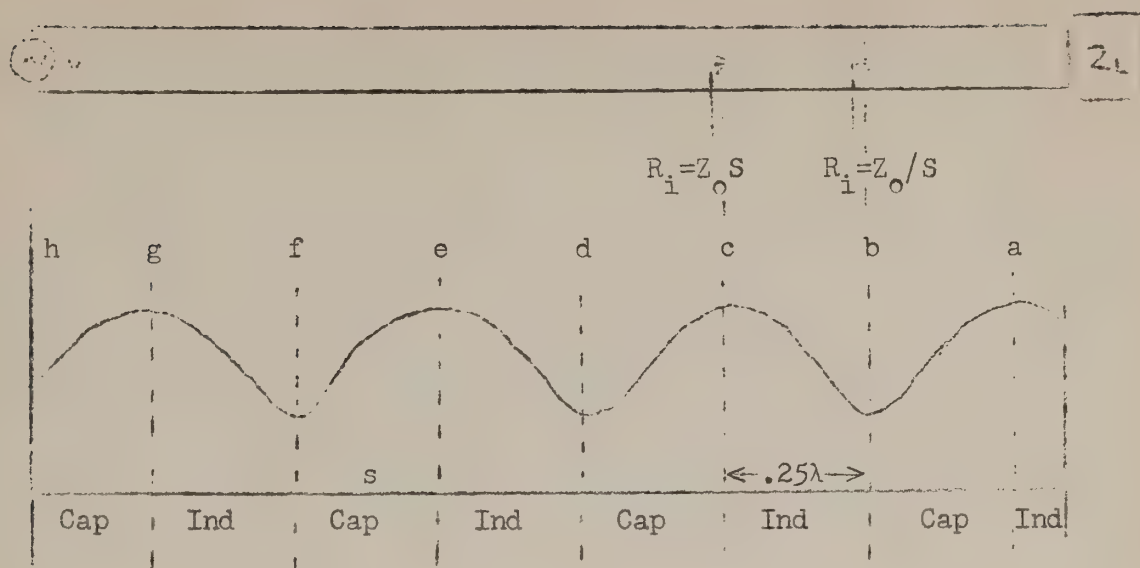


Fig. 2 At points a, c, e and g, $R_i = Z_o S$.
 At points b, d and f, $R_i = Z_o / S$. At point h,
 $Z_i = R_i - j X_i$. The
 input reactance X_i toward the load alternates
 between inductive and capacitive as shown.

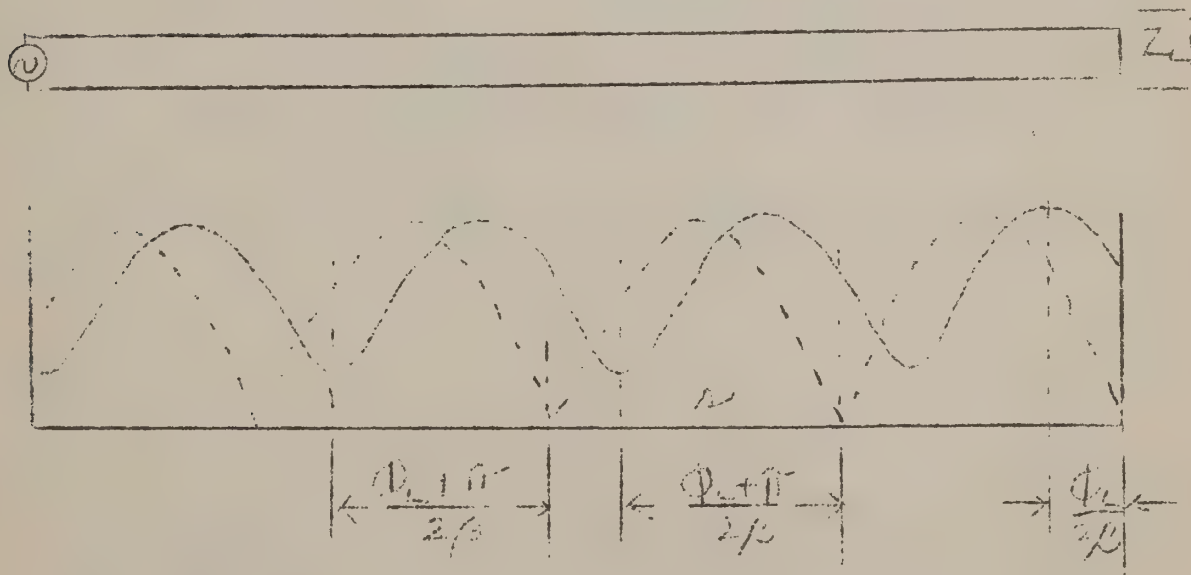


Fig. 3 Illustrating how the angle ϕ_L may be
 obtained from standing wave measurement on a line.

The first minimum, from the load end, occurs when $s = (\phi_L + \pi)/2\beta$. Hence

$$\phi_L = 2\beta s_{\min} - \pi \quad (40)$$

Using this relation it would require an accurate determination of distance, in meters, to the first minimum of V which is seldom possible in practice.

In many cases it is possible to place a short circuit across the line at the load and by a standing wave indicator determine where a minimum due to actual load occurs with respect to the minimum caused by the short circuit. Then the distance s_d in meters between the location of a minimum due to the actual load and the location of the first minimum toward the load end due to a short at the load is given by

$$s_d = \frac{\phi_L + \pi}{2\beta}$$

whence

$$\phi_L = 2\beta s_d - \pi \quad (40a)$$

This is the same as equation 40. Positive values for ϕ_L means the load is inductive reactive and negative values mean capacitive reactive. This method for getting ϕ_L is illustrated by Fig. 3. Note that any two minimums, according to the above rule, may be used. Maximum of the actual standing wave pattern could be used. However, the location of maximums is not as accurate because of their broadness.

Now when $|K_L|$ and ϕ_L are known the load impedance can be determined from the equation

$$\frac{Z_L - Z_0}{Z_L + Z_0} = K_L = |K_L| \angle \phi \quad (41)$$

From which

$$Z_L = Z_0 \frac{1 + |K_L| \angle \phi_L}{1 - |K_L| \angle \phi_L} \quad (42)$$

$$|K_L| = \frac{S-1}{S+1} \quad \text{and} \quad \phi_L = 2\beta s_d - \pi \quad (43)$$

Example. The standing wave ratio on a certain line is 3 and the distance s_d in meters between a minimum due to load Z_L and a minimum when $Z_L = 0$ is such as

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The first part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom. It is shown that the structure of the atom is determined by the laws of quantum mechanics, and that the structure of the atom is determined by the laws of quantum mechanics. The second part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom. It is shown that the structure of the atom is determined by the laws of quantum mechanics, and that the structure of the atom is determined by the laws of quantum mechanics.

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to make βs_d equal to 150° . This gives $\phi_L = 2 \times 150^\circ - 180^\circ = 120^\circ$. Now

$$|K_L| = \frac{3-1}{3+1} = .5$$

Hence

$$K_L = .5 \angle 120^\circ$$

Solving for Z_L gives

$$\begin{aligned} Z_L &= Z_0 \frac{1 + .5(\cos 120^\circ + j \sin 120^\circ)}{1 - .5(\cos 120^\circ + j \sin 120^\circ)} \\ &= Z_0 \frac{1 - .5 \times .5 + j .5 \times .866}{1 + .5 \times .5 - j .5 \times .866} \\ &= Z_0 \times .66 \angle 49.6^\circ \end{aligned}$$

Note 1. s_d in meters is $150^\circ/\beta$. $\beta = 2\pi/\lambda$. Hence s_d in meters = $\frac{150^\circ}{360^\circ} \lambda = \frac{5}{12} \lambda$.
 λ is v/f .

Note 2. $.66 \angle 49.6^\circ$ is the normalized impedance of the load, i.e., normalized to Z_0 the characteristic impedance of the line.

5. Lines of Special Length

a. The Quarter-Wave Length Line. A quarter-wave length line has several interesting and useful properties. It has impedance transforming properties. For example,

$$\begin{aligned} Z_i &= Z_0 \frac{e^{\gamma s} + K_L e^{-\gamma s}}{e^{\gamma s} - K_L e^{-\gamma s}} \\ &= Z_0 \frac{e^{\alpha s} e^{j\beta s} + K_L e^{-\alpha s} e^{-j\beta s}}{e^{\alpha s} e^{j\beta s} - K_L e^{-\alpha s} e^{-j\beta s}} \end{aligned}$$

Now set $s = \lambda/4$ and $K_L = (Z_L - Z_0)/(Z_L + Z_0)$ and get

$$Z_{iq} = \frac{Z_L \sinh \alpha\lambda/4 + Z_0 \cosh \alpha\lambda/4}{Z_L \cosh \alpha\lambda/4 + Z_0 \sinh \alpha\lambda/4} Z_0 \quad (44)$$

For a loss-less line $\alpha\lambda/4 = 0$, $\sinh \alpha\lambda/4 = 0$ and $\cosh \alpha\lambda/4 = 1$. Hence

$$Z_{iq} = \frac{Z_0^2}{Z_L} \quad (45)$$

Thus the impedance Z_L is transformed to Z_0^2/Z_L .

When $Z_L = 0$, i.e., load end is shorted, Z_{iq} becomes very high as seen by the following example.

Referring to equation 44 for $\alpha\lambda/4$ very small and $Z_L = 0$

$$Z_{qs} = Z_0 \coth \frac{\alpha\lambda}{4} = Z_0 \frac{4}{\alpha\lambda}, \text{ i.e., } \coth \frac{\alpha\lambda}{4} = \frac{4}{\alpha\lambda}. \text{ Hence } Z_{qs} = \frac{4Z_0}{\alpha\lambda}.$$

A coaxial line has a characteristic impedance of 50 ohms, an outside diameter of 1-5/8 inches and a 0.67 db loss per 100 feet at 1000 megacycles. This gives $\alpha = \frac{0.67}{8.68} \times \frac{1}{100} \times 30.5 = 2.36 \times 10^{-5}$ nepers per centimeter. At 1000 megacycles $\lambda = 30$ centimeters. Hence $Z_{qs} = 4Z_0/\alpha\lambda = 4 \times 50/2.36 \times 30 \times 10^{-5} = 28.2 \times 10^4$ ohms which is very high but not infinite.

A quarter wave length line open at the far end has a very low input impedance. In this case $Z_{qo} = Z_0 \tanh \frac{\alpha\lambda}{4} = Z_0 \frac{\alpha\lambda}{4}$. Thus for the above line

$$Z_{qo} = \frac{50 \times 2.36 \times 10^{-5} \times 30}{4} = 7.08 \times 10^{-3}$$

which is almost a short circuit.

Another interesting feature of a loss-less quarter wave length line is its use for getting the current in a load impedance that is independent of the load impedance. The relation between a generator voltage and a load current, which is a transfer impedance, is $\frac{E_g}{I_L} = Z_L \cosh \gamma s + Z_0 \sinh \gamma s$. For $\alpha s = 0$ and $\beta s = \pi/2$, $\cosh \gamma s = 0$ and $\sinh \gamma s = j$. Consequently,

$$I_L = -j \frac{E_g}{Z_0} \quad (46)$$

where Z_0 is the characteristic impedance of the quarter wavelength line. Thus it is seen that I_L is independent of Z_L when the load impedance is connected to the generator by a quarter wavelength of line.

b. The Half-Wavelength Line. Examination of the equation for the input impedance of a half wavelength line terminated in an impedance Z_L shows that

$$Z_{ih} = Z_L \quad (47)$$

That is the half wavelength line has no impedance transforming properties.

It is interesting to note an examination of equation 24 shows that the voltage at half wave interval on a loss-less line are equal in magnitude but

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alternate by a phase angle of 180° . Thus if V_L is the voltage at the load $V_1 = -V_L$ is the voltage $\frac{1}{2} \lambda$ from the load, $V_2 = V_L$ 1λ from the load and so on. Also impedances connected across the line at half wave length intervals will all be driven by the same voltage (in magnitude) but the phase will alternate by 180° .

6. Reflection Coefficient and the Smith Chart

The magnitude and angle of the load reflection coefficient K_L depends upon the relationship between Z_L and Z_0 and is given by

$$K_L = \left| \frac{Z_L - Z_0}{Z_L + Z_0} \right| \angle \phi_L$$

where

$$\phi_L = \tan^{-1} \frac{X_L}{R_L - R_0} - \tan^{-1} \frac{X_L}{R_L + R_0}$$

for a lossless line because $Z_0 = R_0 + j 0$.

Now at any point on a lossless line where the input impedance toward the load is not equal to Z_0 there is a reflection coefficient K_i because of mismatch of impedances. Previously it was shown that $|K_L| = (S-1)/(S+1)$ where S is the standing wave ratio. Thus the mismatch at the load fixes the standing wave ratio and $|K_i| = |K_L|$. However the angle of K_i is not the same as that of K_L . The input impedance toward the load is, from Eq. 34

$$Z_i = Z_0 \frac{e^{j\beta s} + K_L e^{-j\beta s}}{e^{j\beta s} - K_L e^{-j\beta s}} = Z_0 \frac{1 + |K_L| \angle \phi_L - 2\beta s}{1 - |K_L| \angle \phi_L - 2\beta s}$$

From which

$$K_i = \frac{Z_i - Z_0}{Z_i + Z_0} = |K_L| \angle \phi_L - 2\beta s \quad (48)$$

Thus the magnitude of K_i remains equal to the magnitude of K_L but the angle changes by $-2\beta s$. This is expected because at the end of a half wave-length, $2\beta s = 2\pi$, the input impedance is Z_L and the angle of K_i here is the same as at the load.

The absolute magnitude of K lies between 0 and 1. The locus of K_i in a polar diagram is a circle whose radius is the magnitude of K_L . The angle of K_i , i.e., $\phi_i = \phi_L - 2\beta s$, is the angle subtended by the phasor K_i and the u axis to the right of center. This is illustrated by Fig. 4.

1. The first part of the paper is devoted to a general discussion of the problem of the existence of solutions of the system of equations

which is the system of equations of the theory of the motion of a particle in a magnetic field.

2. The second part of the paper is devoted to a detailed analysis of the case of a uniform magnetic field.

3. The third part of the paper is devoted to a detailed analysis of the case of a non-uniform magnetic field.

4. The fourth part of the paper is devoted to a detailed analysis of the case of a magnetic field with a gradient.

5. The fifth part of the paper is devoted to a detailed analysis of the case of a magnetic field with a gradient and a uniform field.

6. The sixth part of the paper is devoted to a detailed analysis of the case of a magnetic field with a gradient and a uniform field.

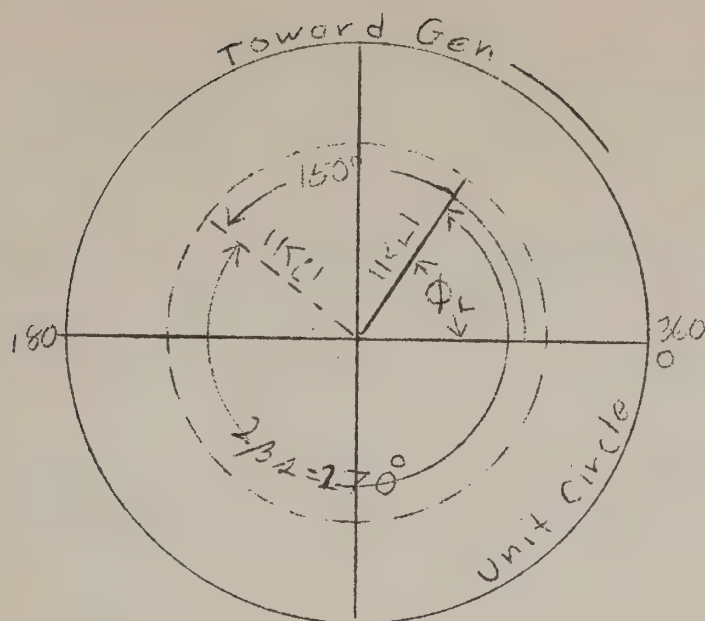


Fig. 4 $K_L = .7 \angle 60^\circ$
 At 135° from the load
 $K_i = .7 \angle 60-270^\circ = .7 \angle -210^\circ = .7 \angle 150^\circ$

Since the magnitude of K_L , or K_i , is $(S-1)/(S+1)$ obviously a scale may be constructed for the unit circle of Fig. 4 that will give S for each magnitude of K . For example, for $K_L = .7$, $S = 1 + K_L / (1 - K_L) = 1.7/.3 = 5.66$. The scale for S will range from 1 to ∞ , will be contained within the unit radius of K circle and will not be uniform. Now

$$K_i = \frac{Z_i - Z_o}{Z_i + Z_o} = \frac{(Z_i/Z_o) - 1}{(Z_i/Z_o) + 1} = \frac{z_i - 1}{z_i + 1} \quad (49)$$

$$= \frac{r_i - 1 + j x_i}{r_i + 1 + j x_i} \quad (50)$$

r_i and x_i are called normalized input resistance and reactance, i.e., they are normalized to the characteristic impedance of the transmission line. Set

$$K_i = \frac{r_i - 1 + j x_i}{r_i + 1 + j x_i} = u + j v \quad (51)$$

solve for u and v and get

$$u = \frac{r_i^2 - 1 + x_i^2}{(r_i + 1)^2 + x_i^2} \quad (52)$$

1. The first part of the paper is devoted to a general discussion of the problem of the existence of solutions of the system of equations

$$\frac{dx}{dt} = A(x)u, \quad \frac{dy}{dt} = B(x)y,$$

where $A(x)$ and $B(x)$ are matrices depending on x .

2. In the second part, we consider the case where $A(x)$ and $B(x)$ are constant matrices.

3. In the third part, we consider the case where $A(x)$ and $B(x)$ are functions of x and y .

4. In the fourth part, we consider the case where $A(x)$ and $B(x)$ are functions of x and y and t .

5. In the fifth part, we consider the case where $A(x)$ and $B(x)$ are functions of x and y and t and u .

$$v = \frac{2x_i}{(r_i+1)^2 + x_i^2} \quad (53)$$

$$\text{These result in} \quad \left(u - \frac{r_i}{r_i+1}\right)^2 + (v - 0)^2 = \left(\frac{1}{r_i+1}\right)^2 \quad (54)$$

$$\text{and} \quad (u-1)^2 + \left(v - \frac{1}{x_i}\right)^2 = \frac{1}{x_i^2} \quad (55)$$

These equations represent two families of circles. Circles centered along the u axis are constant r_i circles and those along the v axis constant x_i circles. Since $\sqrt{u^2 + v^2} = |K_i|^2 \leq 1$ the whole range of normalized values are contained within a unit circle and a plot of the r_i and x_i circles becomes a reflection coefficient chart commonly known as a Smith Chart illustrated by Fig. 5.

Because of the fact that r_i and x_i are repeated every half wave length one revolution around the chart represents 180° . This differs from the circle of Fig. 4, which has 360° for one revolution. Portions of circles of constant inductive reactances, or capacitive susceptances are found in the upper half of the chart and capacitive reactances, or inductive susceptances in the lower half. Resistances, or conductances, are indicated along the u axis for the circles that have their center on the u axis.

The magnitude of the reflection coefficient is equal to the distance in centimeters from the center of the unit circle to the location of r_L and x_L divided by the radius of the unit circle in centimeters. The angle of the reflection coefficient is the geometric angle between a line drawn from the center of the unit circle to the location of r_i and x_i and the u axis to the right of center. When using conductance g_i and susceptance b_i the reference is on the u axis to the left of center. In either case if the location of r_i , or g_i , and x_i or b_i is in the upper half of the chart ϕ_i is positive and if in the lower half ϕ_i is negative, or if positive angles are always preferred subtract the latter angle from 360° . Note that the geometric angle is specified in the above rules. Angles determined from the scale on the rim of the Smith Chart must be multiplied by 2 to get the angle of the reflection coefficient.

A circle with center $u = 0$ and passing through r_i and x_i , i.e., the K circle, is also a constant standing wave ratio circle. Standing wave ratios, always greater than one, may be read on the resistance, or conductance, scale on the u axis to the right of the center. This is so because, for example, when $x_i = 0$ and $r_i = 3$ the standing wave ratio is 3, or S is equal to the normalized resistance r for $r > 1$, or $1/r$ for $r < 1$.

1. The first part of the paper is devoted to a general discussion of the problem of the existence of solutions of the system of equations

which are satisfied by the functions $u_i(x, y, z)$ and $v_i(x, y, z)$ in the domain D of the space E_3 . The functions u_i and v_i are assumed to be continuous in D and to satisfy the boundary conditions

on the boundary S of the domain D . The functions u_i and v_i are also assumed to satisfy the conditions

where \mathbf{r} is the vector of the position of the point (x, y, z) in the domain D and \mathbf{n} is the vector of the normal to the boundary S at the point (x, y, z) . The functions u_i and v_i are also assumed to satisfy the conditions

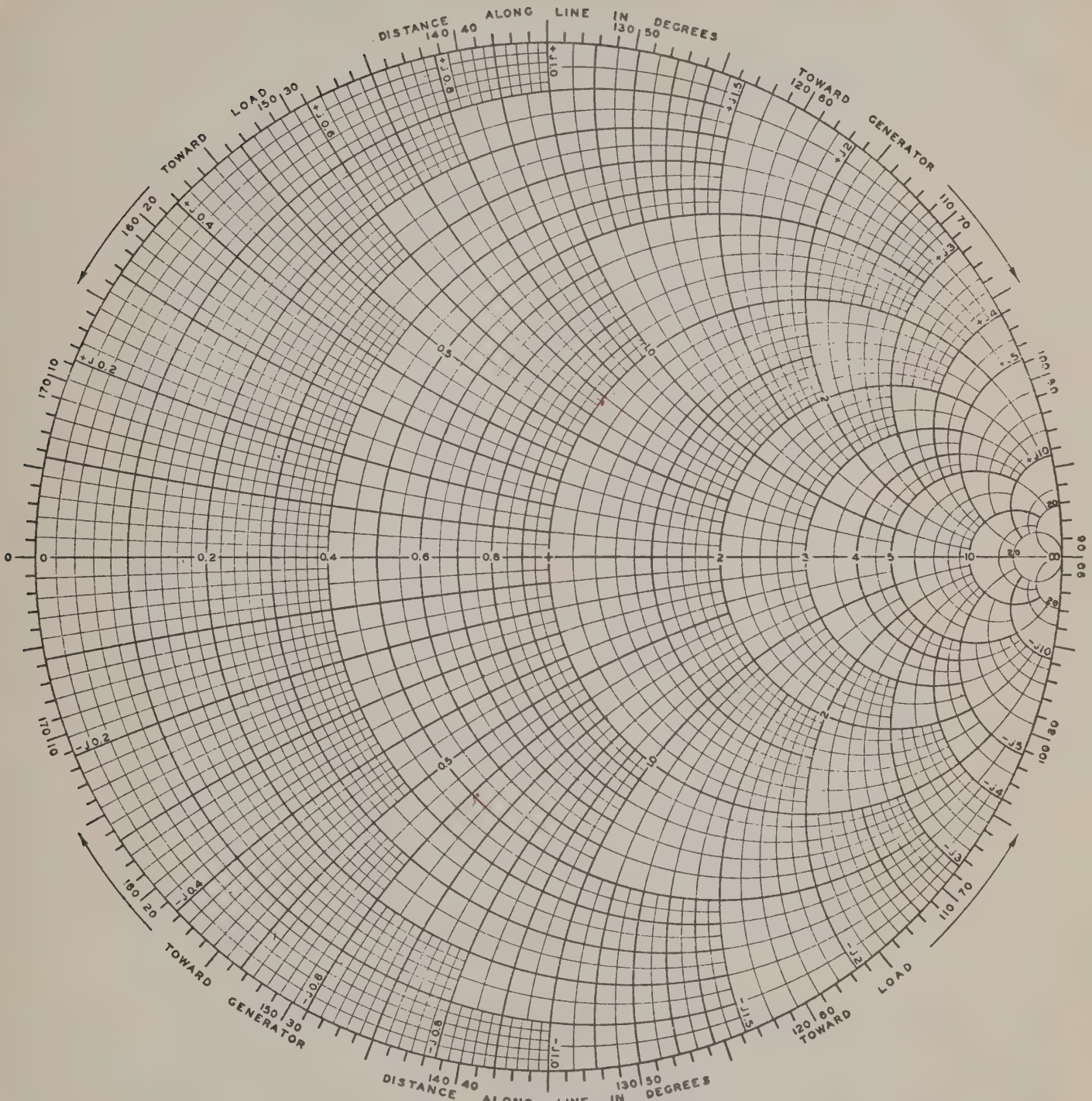
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REFLECTION COEFFICIENT CHART

CONTOURS ON CHART ARE RECTANGULAR COMPONENTS OF IMPEDANCE OR ADMITTANCE



360° ALONG LINE IS ONE WAVELENGTH



1 C.

1 C.

1 C.

Examples on use of the Smith Chart

Example 1. Given a transmission line $5/8$ wave length long with a characteristic impedance of $50 + j0$ ohms terminated in a load impedance of $100 - j100$ ohms. By the Smith Chart determine the following:

- a. magnitude and angle of \underline{K}
- b. the standing wave ratio S
- c. the input impedance

Procedure: Refer to Fig. 6 for Solution

- a. The normalized load impedance is $(100 - j100)/50 = 2 - j2$ (P_1)
- b. The point $2 - j2$ is located as P_1 on the chart and a radial line is drawn from the center through this point to the rim of the chart. Twice the chart degrees from the 90° mark going counter clockwise to the point where this radial line cuts the rim of the chart is the angle ϕ of \underline{K} . This angle is $2 \times 165.2^\circ = +330.4^\circ$, or -29.6° .
- c. A constant \underline{K} circle (the same as a constant S circle) is drawn passing through the point $2 - j2$. The ratio of radius of this circle to the radius of the unit circle gives the magnitude of \underline{K}_L . This ratio is 0.62 . Hence $\underline{K}_L = 0.62 \angle 330.4^\circ$.
- d. The standing wave ratio is read from the scale of normalized resistance, or conductance along the right hand portion of the u axis. This is 4.26 .
- e. The normalized input impedance is found by starting at $2 - j2$ and moving around the constant S circle $5/8^\lambda$ or 225 chart degrees toward the generator. This is once around $+45$ chart degrees more ($225^\circ - 180^\circ = 45^\circ$). This gives $z_i = 0.31 - j0.54$ which is P_2 . Hence $Z_i = (0.31 - j0.54)50 = 15.5 - j27$. The reflection coefficient \underline{K}_i at the input is $0.62 \angle 240.4^\circ$. Note the magnitude of \underline{K}_i is same as that of \underline{K}_L but the angle is different.

Example 2. Suppose the line of example 1 is shunted by 125 ohms at $1/12$ of a wave length (30°) from the load.

- a. What is the input impedance at $5/8$ wave length from the load?
- b. What is the magnitude and angle of the reflection coefficient at $1/12$ wave length from the load after the 125 ohm resistor is shunted across the line?
- c. What are the standing wave ratios on the two sections of the line?

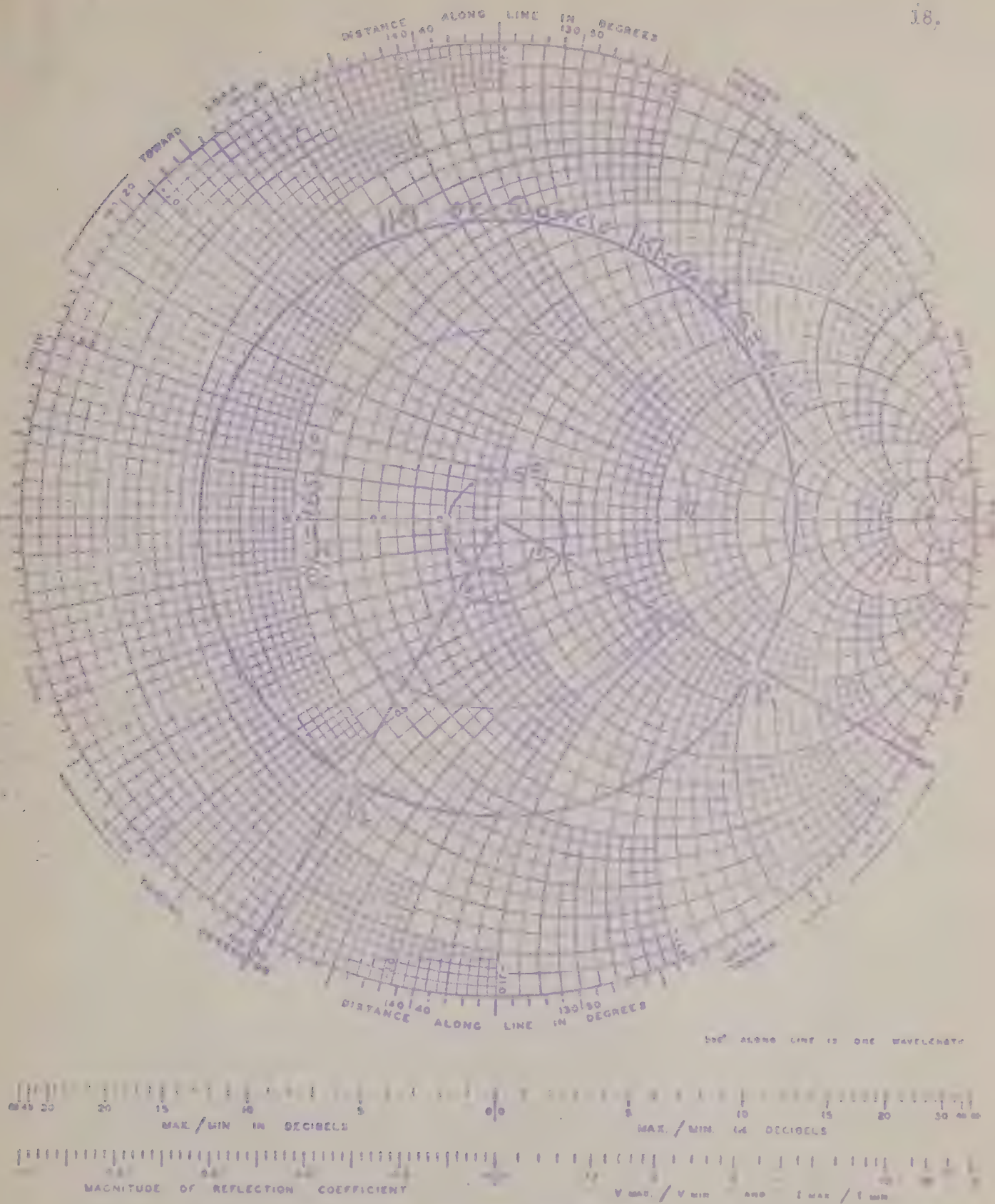


FIG. 2. TRANSMISSION LOSS COEFFICIENT FOR REFLECTOR 1

The transmission loss coefficient is plotted at point P_1 . The corresponding input impedance is found at point P_2 which is 225 chart degrees toward the generator from P_1 .

Procedure: Refer to Fig. 7 for Solution.

- a. First locate the normalized load impedance $2-j2$ on the chart as P_1 .
- b. Draw a standing wave ratio circle S_1 through this point. Then travel 30 chart degrees toward the generator on this circle. This locates the normalized input impedance at the point P_2 where the 125 ohm resistor is to be shunted across the line.
- c. Since the resistor is being shunted across the line it will be in parallel with the input admittance of the line. Consequently convert from normalized input impedance to normalized admittance. This is done on the chart by locating a point P_3 on the constant standing wave ratio circle but displaced 90 chart degrees from the point P_2 located in b. (see footnote 1 for the justification on this procedure)
- d. Read the input conductance of the line which is 0.45. Convert 125 ohms to a normalized conductance, which is 0.4. The $0.45+0.4 = 0.85$ is the normalized conductance (line plus 125 ohm resistor in parallel). The susceptance remains unchanged and is 0.89 on the chart.
- e. Now locate the admittance $0.85 + j0.89$ as point P_4 and draw a standing wave ratio circle S_2 through this point. Then travel toward the generator $225^\circ - 30^\circ = 195^\circ$ on this new standing wave ratio circle to P_5 . This gives a normalized admittance of $1.45 + j1.1$. Convert to normalized impedance by the method given in c. This gives $Z_i = (0.44 - j0.34)50 = 22 - j17$ ohms at $5/8$ wave length from the load with 125 ohms bridged across the line 30° from the load.
- f. The reflection coefficients at the location of the 125 ohm resistor is
 - (1) $0.62 \angle 270.4^\circ$ before the 125 ohm resistor is shunted across the line.
 - (2) $0.45 \angle 254^\circ$ after the 125 ohm resistor is shunted across the line.

The reflection coefficient at the input is $K_i = 0.45 \angle 224^\circ$.

- g. The standing wave ratio S on the line between load and 125 ohm resistor is 4.26. Between the input and 125 resistor $S = 2.6$. Both of these values are read on the right portion of the u axis where the standing wave circles cross this axis.

Note 1. The justification for the conversion from normalized impedance to normalized admittance as stated in c is as follows: moving from the impedance point through 90 chart degrees along a constant standing wave

1. The first part of the paper discusses the importance of maintaining accurate records of all transactions.

2. It then goes on to describe the various methods used to collect and analyze data, including interviews, surveys, and focus groups.

3. The next section discusses the results of the study, which show that there is a significant correlation between the variables.

4. This is followed by a discussion of the implications of the findings for practice and policy, and a conclusion.

5. The paper also includes a list of references and an appendix containing the survey instrument used in the study.

6. Finally, the author acknowledges the limitations of the study and suggests areas for future research.

7. The paper is written in a clear and concise style, and is well organized and easy to read.

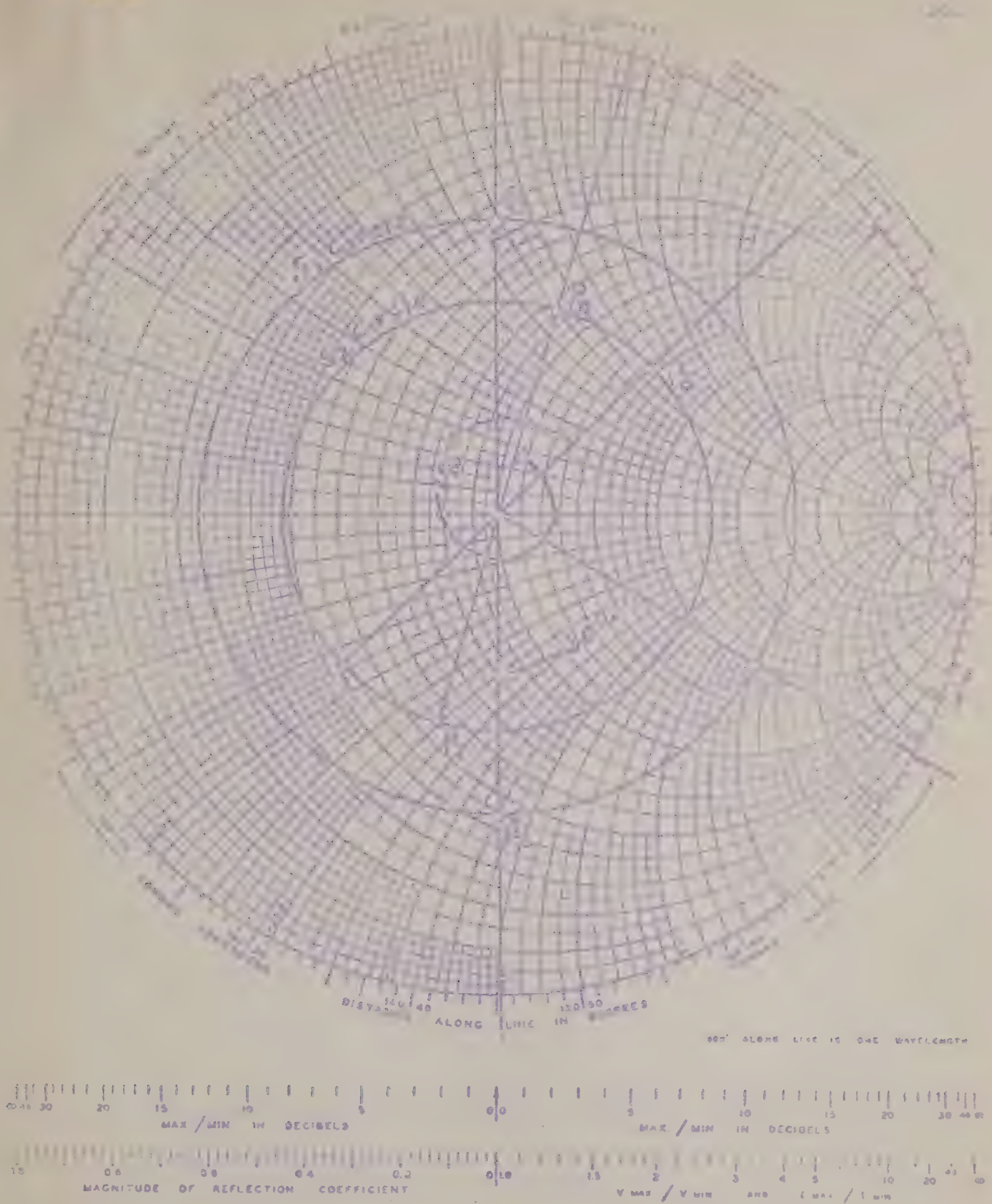


Fig. 7 ILLUSTRATING THE SOLUTION FOR EXAMPLE 2.

For P_1 $z_L = 2 - j2$, for P_3 30° from load $y_1 = 0.45 + j0.89$, for P_4
 $z_1 + \frac{1}{125} \times 50 = 0.4 + 0.45 + j0.89$ and finally for P_6 225° from load
 $z_{in} = (0.44 - j0.34)50 = 22 - j17$ ohms. P_7 is not needed in this solution.
 Note y for $P_3 = 1/(z$ for $P_2)$ and z for $P_6 = 1/(y$ for $P_5)$.

ratio circle to the corresponding admittance point is equivalent to inserting a quarter wave line between these two points. Now the input impedance of a quarter wave line terminated in Z_L is $Z_{iq} = Z_c^2/Z_L = Z_c^2 Y_L$ in normalized notation $z_{iq} = Z_c Y_L$ and $y_L = Z_c Y_L$. Hence $z_{iq} = y_L$. Consequently on the chart z_L is converted to y_L by travelling 90 chart degrees around the standing wave ratio circle.

Note 2. In example 2 part d it is found that placing a resistor in shunt with the line changes only the conductance at the point of application. The susceptance remains unchanged. Now if a susceptance, in place of a conductance, were placed across the line the conductance would remain unchanged and point P_3 would move to a new point P_4 on a constant conductance circle.

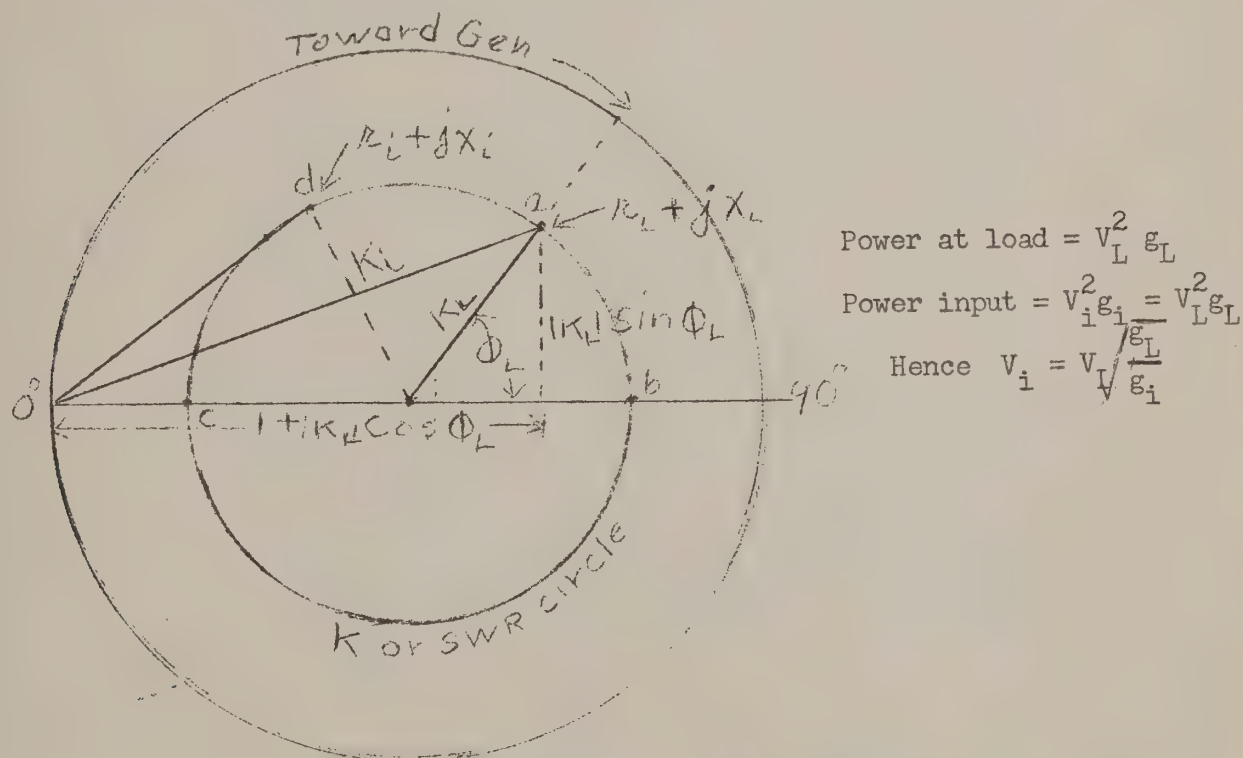


Fig. 8 Illustrating the crank diagram principle for getting the voltage versus distance on a transmission line

$$V_i = V_L \frac{0-d}{0-a}$$

Note 1. When admittances are plotted the reference point for V is the 90° mark on the chart.

Note 2. ϕ_L shown is the geometric angle which is twice the chart angle.



7. Voltages from Smith Chart (Crank Diagrams)

Referring to a previous equation the change in voltages, with respect to s , on a loss-less transmission line is given by $1 + |K_L| \cos(\phi - 2\beta s) + j|K_L| \sin(\phi - 2\beta s)$ which is located on a constant K circle as shown in Fig. 8. Then if line $O-a$ represents the load voltage, line $O-d$ will represent the voltage at any location βs degrees from the load. Therefore if $O-a$ represents V_L volts then

$$V(\text{at } \beta s \text{ degrees from load}) = V_L \frac{O-d}{O-a}$$

Note 1. $O-b$ represents the maximum voltage on the line and $O-c$ represents the minimum voltage. Hence $(O-b)/(O-c) = S$ the standing wave ratio.

Note 2. When normalized admittance is plotted instead of normalized impedance the reference point from which voltage is scaled is at the 90° mark on the chart.

Note 3. When the magnitude of the reflection coefficient is changed by adding shunt impedance at any point on the line as in example 2 the procedure for getting voltages is a bit more complicated. On the Smith Chart for example 2 let the distance O to P_1 represent a given voltage. Then the distance from O to P_2 will represent the voltage where the resistor is shunted across the line. The distance from O to P_7 represents the same voltage found at P_2 . P_7 is the impedance corresponding to the admittance found at P_4 . In other words a change of voltage scale takes place at the location of the resistor. Thus for example, if the length of line $O-P_2$ represents 10 volts then the length of line $O-P_7$ also represents 10 volts. Finally the distance O to P_6 multiplied by the new scale is the voltage at the input. In other words the voltage at the input is given by

$$V_i = V_L \frac{O-P_2}{O-P_1} \times \frac{O-P_6}{O-P_7}$$

Note: The above equation for V_i does not hold when impedance is added in series with the line. This case is usually not important.



Transmission Line Theory. (General)

1. Introductory

The transmission lines treated in these notes consists of two conductors, usually copper or aluminum, arranged in various configurations some of which are illustrated symbolically by Fig.1. The purpose of a transmission line is to guide electrical energy in some form, such as 60 cycle energy for home and commercial purposes or modulated energy in the form of telephone conversation and others, from one location to another. The conductors of the transmission line do not transmit the energy but serve only as guides for the energy which is transmitted in the electromagnetic field surrounding the conductors. This is the only concept that satisfies the Maxwell field equations and agrees with other systems of energy transmission such as hollow wave guides where only one conductor exists and radio transmission where there are no conductors.

The mathematical treatment and physical behavior of a transmission line depends upon whether the line is electrically short or long. Actual physical length and electrical length are two different quantities when deciding whether a line is short or long. A transmission line is 100 meters long physically. At 60 cycles per second it is very short electrically. At 10^8 cycles per second it is very long electrically. Electrical length is expressed in terms of wavelength for a particular operating frequency. Wave length is the distance for which a voltage, or current, undergoes a 360° phase shift from one end of the line to the other. One wave length at 60 cps is 3100 miles. Whereas it is only 3 meters at 10^8 cps.

A transmission line is a linear circuit consisting of resistance R, inductance L, capacitance C and leakage conductance G uniformly distributed throughout the length of the line. The resistance and inductance are in series with the line conductors and the capacitance and leakage conductance are in shunt as illustrated in Fig. 2. Because of the distributed nature of these parameters the current and voltage on a line undergo continual changes in magnitude and phase along the line in relation to the current and voltage at some reference point such as the sending end or the receiving end.

A very short, electrically, length of transmission line may be represented as a T or π network as shown in Fig. 3. For example 50 miles of 60 cycle power line is only .0155 wave lengths long and may therefore be represented as a single T or π network. For a power line only a few miles long it is generally sufficient

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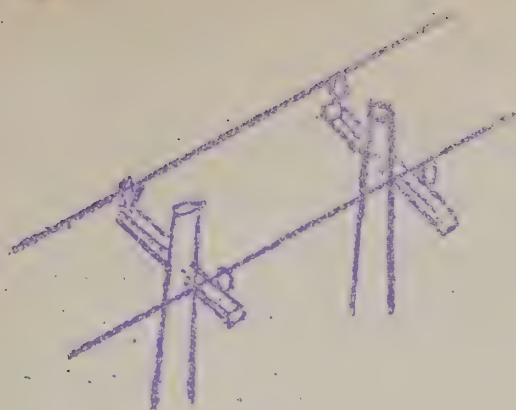
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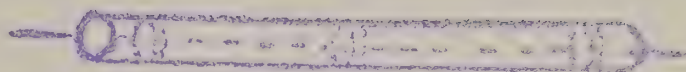
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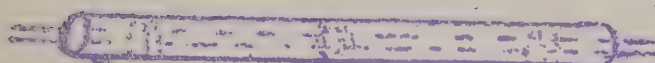
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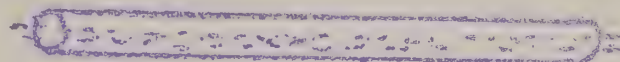
a. open wire line



b. coaxial line



c. shielded pair line
outside shield usually
copper and grounded



d. cable type line
sheath may be metallic
some cables contain
several pairs of conductors

Fig. 1. Several Types of Transmission Lines

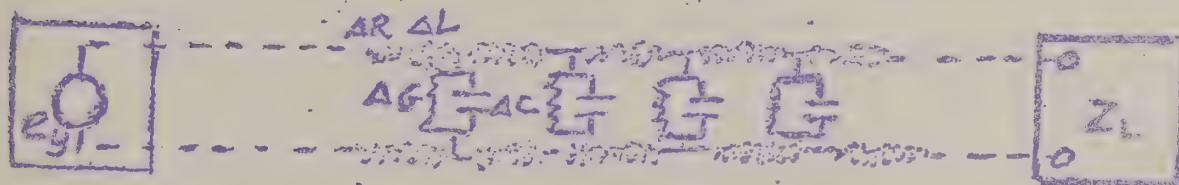


Fig. 2. A very short piece of transmission line between the source e_g and the load Z_L showing how R, L, C and G are disposed to form a circuit of uniformly distributed parameters.

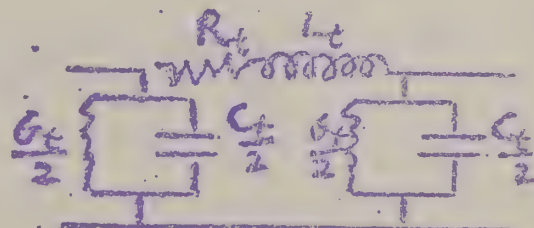
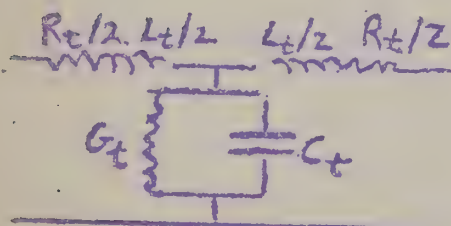


Fig. 3. A very short transmission line represented by a T or π network. The parameters $R_t, L_t, C_t,$ and G_t are for the entire length of line.

to replace the line by its overall resistance and inductance and neglect its capacitance and leakage conductance.

When a line becomes of the order of .05 wave lengths or longer and there is only one frequency of interest the line may be represented by a single T or π network whose parameters are determined from the open and short circuit impedances of the line. However, open and short circuit impedances must be determined either from measurements or from mathematical theory which takes into consideration the uniform distribution of the line parameters. Thus for a line that cannot be represented by a T or π network without the use of the open and short circuit impedance it becomes simpler to treat the line as a uniformly distributed parameter circuit and derive and solve the differential equation which apply.

2. Derivation of the Differential Equations for the Transmission Line.

In order to arrive at a pair of equations which express the relations for voltage and current at any point on a transmission line it is necessary to first derive the fundamental differential equations for the line. These equations are then solved for voltage and current in terms of the line parameters, the distance to a point in question and the terminal conditions.

The differential equations are developed by applying the e.m.f. and current laws to an infinitesimal portion of the line. Fig. 4 represents an infinitesimal portion of a transmission line. Since the current in one conductor is equal to, but opposite in direction, at any instant of time to the current in the other conductor, all line resistance and inductance of the infinitesimal portion of line is placed in the upper branch as shown.

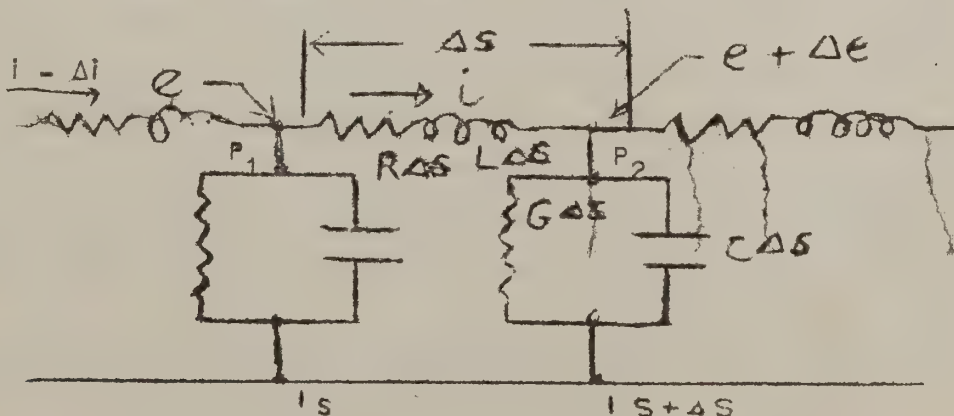


Fig. 4 An infinitesimal length of transmission line. The parameters R , L , C and G are for a unit length of line.

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Referring to Fig. 4 the difference in potential between the line conductors at point P_2 is equal to the potential difference e at P_1 plus Δe where Δe is the incremental change in potential difference in going from P_1 to P_2 because of the current i in RAs and LAs. In equation form this becomes

$$e - (Ri + L \frac{\partial i}{\partial t}) \Delta s = e + \Delta e$$

Which gives, as $\Delta s \rightarrow 0$

$$\frac{\partial e}{\partial s} = - (Ri + L \frac{\partial i}{\partial t}) \quad (1)$$

In the infinitesimal portion of line from P_1 to P_2 the current i and potential e are changing with respect to both time and distance, thus the partial derivatives.

In a similar manner as the above the current to the right of P_2 differs from the current between P_1 and P_2 by Δi where Δi is the incremental change in current due to the shunt paths GAs and CAs. In equation form

$$i = \Delta i - \Delta i - (Ge + C \frac{\partial e}{\partial t}) \Delta s$$

which gives, as $\Delta s \rightarrow 0$

$$\frac{\partial i}{\partial s} = - (Ge + C \frac{\partial e}{\partial t}) \quad (2)$$

Equations 1 and 2 are the fundamental differential equations for a transmission line. The equations give the space rate-of-change of e and i respectively, and at each point on the line e and i are changing with respect to time. This indicates that e and i are propagated along the line in a wave pattern. This will become clearer when the differential equations are solved and physical interpretation are given for the resulting solutions.

Solutions of equations 1 and 2 are best carried out by first obtaining two equations, each of which is expressed in terms of a single dependent variable, i.e. one equation containing only e and the other containing only i . This is carried out as follows.

Differentiation of 1 with respect to s gives

$$\frac{\partial^2 e}{\partial s^2} = - (R \frac{\partial i}{\partial s} + L \frac{\partial^2 i}{\partial t \partial s}) \quad (3)$$

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Differentiation of 2 with respect to t gives

$$\frac{\partial^2 i}{\partial t \partial s} = (G \frac{\partial e}{\partial t} + C \frac{\partial^2 e}{\partial t^2}) \quad (4)$$

Now substitute $\frac{\partial i}{\partial s}$ from equation 2 and $\frac{\partial^2 i}{\partial t \partial s}$ from equation 4 into equations 3 and get

$$\frac{\partial^2 e}{\partial s^2} = RGe + (RC + LG) \frac{\partial e}{\partial t} + LC \frac{\partial^2 e}{\partial t^2} \quad (5)$$

In like manner

$$\frac{\partial^2 i}{\partial s^2} = RG i + (RC + LG) \frac{\partial i}{\partial t} + LC \frac{\partial^2 i}{\partial t^2} \quad (6)$$

These are the second order differential equations that govern the behavior of e and i for the transmission line.

Except for the special case which is the steady state solution when e and i are sinusoidal functions of time the solutions of equations 5 and 6 are quite complicated both mathematically and physically. Hence in order to acquire some physical insight into the nature of e and i before proceeding with steady state sinusoidal solutions equation 5 and 6 will be modified to represent the loss-less line condition, i, e, $R = G = 0$. In this sense they become

$$\frac{\partial^2 e}{\partial s^2} = LC \frac{\partial^2 e}{\partial t^2} \quad (7)$$

$$\frac{\partial^2 i}{\partial s^2} = LC \frac{\partial^2 i}{\partial t^2} \quad (8)$$

3. Travelling Waves.

Equations 7 and 8 are generally known as the wave equations for loss-less transmission lines. Just why these equations portray wave motion may be seen by examining in some detail their steady state solutions when e and i are sinusoidal functions of time. An assumed solution for equation 7 is

$$e = e_1 \cos(\omega t - \beta s) + e_2 \cos(\omega t + \beta s) \quad (9)$$

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That this is a solution may be shown as follows:

$$\frac{\partial e}{\partial s} = + \beta e_1 \sin (\omega t - \beta s) - \beta e_2 \sin (\omega t + \beta s)$$

$$\frac{\partial^2 e}{\partial s^2} = - \beta^2 e_1 \cos (\omega t - \beta s) - \beta^2 e_2 \cos (\omega t + \beta s)$$

$$= \beta^2 [-e_1 \cos (\omega t - \beta s) - e_2 \cos (\omega t + \beta s)]$$

Likewise

$$\frac{\partial^2 e}{\partial t^2} = - \omega^2 e_1 \cos (\omega t - \beta s) - \omega^2 e_2 \cos (\omega t + \beta s)$$

$$= \omega^2 [-e_1 \cos (\omega t - \beta s) - e_2 \cos (\omega t + \beta s)]$$

Then if $\beta = \omega \sqrt{LC}$ equation 9 becomes a solution of equation 7. Now by graphing the first term of equation 9 in Fig. 5 for several instants of time it is seen that this term represents a wave moving in the positive direction of s . In a similar manner the second term of equation 9 represents a wave moving in the negative s direction. For one complete period, i.e. $1/\text{frequency}$, the wave moves a distance called one wave length λ . Thus

$$s_\lambda = \lambda = \frac{2\pi}{\beta} \quad \text{OR} \quad \beta = \frac{2\pi}{\lambda} \quad (10)$$

Now s = velocity of wave multiplied by time or

$$s_\lambda = \lambda = \text{vel.} \times t = \text{vel.} / f = \frac{2\pi}{\beta}$$

$$\text{Hence} \quad \text{vel} = \frac{2\pi f}{\beta} \quad (11)$$

$$\text{and} \quad \beta = 2\pi f \sqrt{LC} \quad (12)$$

Consequently velocity = $\frac{1}{\sqrt{LC}} = v_p$ and is called the phase velocity of the wave. For a transmission line with air or vacuum dielectric it may be shown that if the self inductance due to magnetic flux linkages inside the conductors is neglected $1/\sqrt{LC} = c$ the velocity of light in free space. For any lossless line $v_p = c/\sqrt{\epsilon_r}$ where ϵ_r is the dielectric constant of the medium around the line conductors.

1. The first part of the paper is devoted to a general discussion of the problem.

2. The second part is devoted to a detailed analysis of the case of a single particle.

3. The third part is devoted to a detailed analysis of the case of a system of particles.

4. The fourth part is devoted to a detailed analysis of the case of a system of particles.

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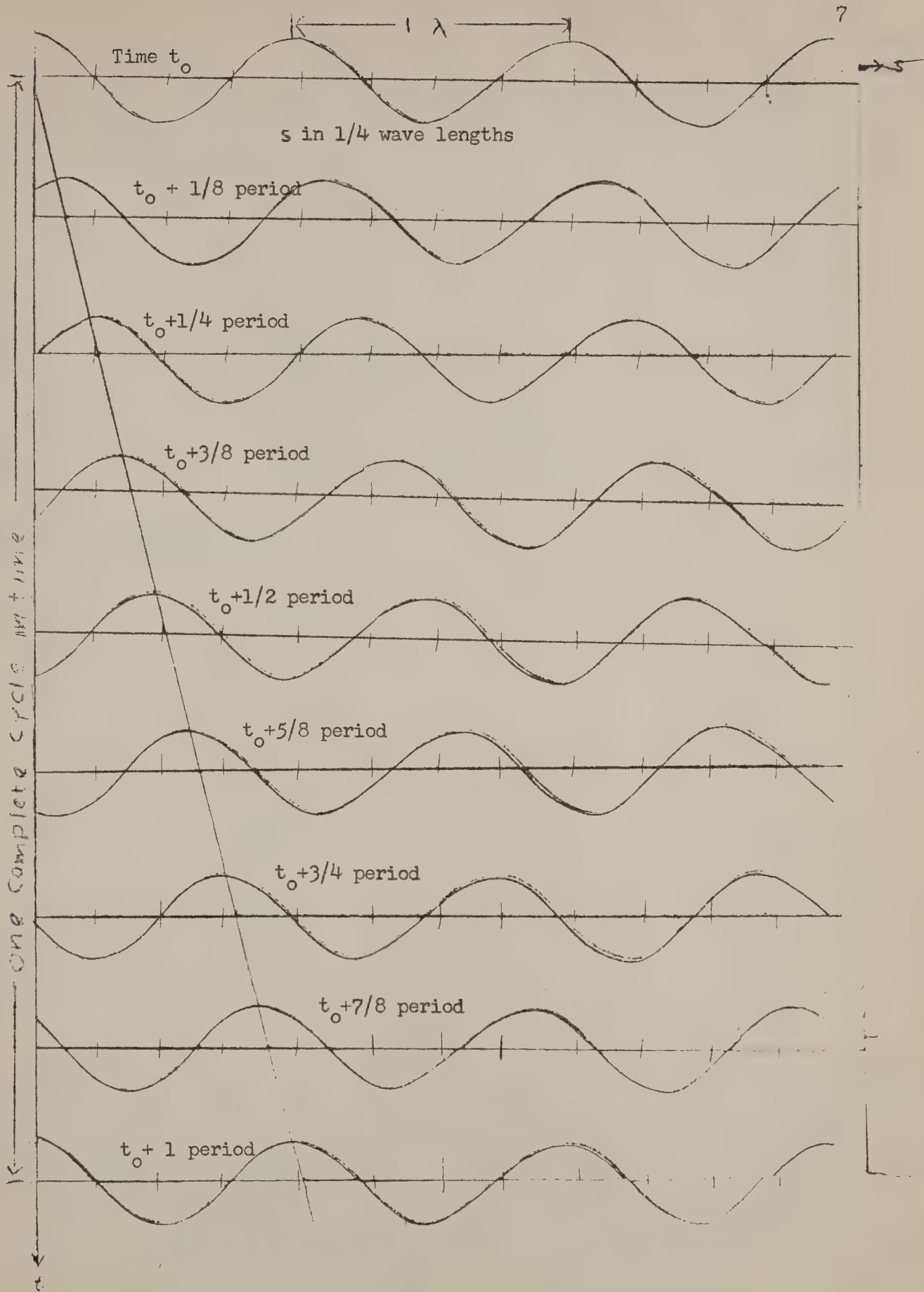


Fig. 5. Wave moves to right 1λ in 1 cycle of time

Assume now a line so terminated that only the first term of equation 9 exists. This is possible as will be seen later. Suppose two oscilloscopes were connected across the line s meters apart. Both oscilloscopes would show sinusoidal voltage - time waves. However, the voltage-time wave shown on the oscilloscope farthest from the source end would be βs degrees lagging the voltage-time wave shown by the other oscilloscope. Suppose the oscilloscope nearest the source were connected to the line at the instant the voltage was e_a volts and then moved along the line at a velocity v_p . The oscilloscope would continue to show e_a volts. These observations help to show that a travelling wave exists on the line.

When the resistance and leakage conductance are not zero, as is the case for a realizable line, the travelling waves suffer attenuation along the line. This will be shown later when the equations for the general case are discussed.

4. Transient considerations.

Returning now to the basic differential equations namely

$$\frac{\partial e}{\partial s} = -(Ri + L \frac{\partial i}{\partial t})$$

$$\frac{\partial i}{\partial s} = -(Ge + C \frac{\partial e}{\partial t})$$

and setting $R = G = 0$ for the loss less case there results

$$\frac{\partial e}{\partial s} = -L \frac{\partial i}{\partial t} \quad (13)$$

$$\frac{\partial i}{\partial s} = -C \frac{\partial e}{\partial t} \quad (14)$$

A general solution of equation 13 for e is

$$e = e_1 f(t - \frac{s}{v}) + e_2 f(t + \frac{s}{v}) \quad (15)$$

where $v = 1/\sqrt{LC}$ and is the velocity of propagation as was found for the steady state solution when e and i vary sinusoidally with time. It may be shown from fundamental dimension that $1/\sqrt{LC}$ is velocity.

Now examining, in detail the first term of equation 15 it is seen that

1. The first part of the paper discusses the importance of the study and the objectives of the research.

2. The second part of the paper describes the methodology used in the study, including the data collection and analysis techniques.

3. The third part of the paper presents the results of the study, which show a significant positive correlation between the variables.

4. The fourth part of the paper discusses the implications of the findings and provides recommendations for future research.

5. The fifth part of the paper concludes the study and summarizes the main findings.

References

1. Smith, J. (2010). The impact of social media on communication. *Journal of Communication*, 40(1), 1-15.

$$\begin{aligned}\frac{\partial e}{\partial s} &= e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \frac{\partial(t - \frac{s}{v})}{\partial s} \\ &= e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \left(-\frac{1}{v}\right)\end{aligned}\quad (16)$$

Assume now the corresponding solution for i , i.e.

$$i = \frac{e_1}{R_o} f(t - \frac{s}{v}) - \frac{e_2}{R_o} f(t + \frac{s}{v}) \quad (17)$$

and taking the partial derivative of the first term, i.e.

$$\frac{\partial i}{\partial t} = \frac{e_1}{R_o} \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})}$$

Then

$$e_1 \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \left(-\frac{1}{v}\right) = -L \frac{e_1}{R_o} \frac{df(t - \frac{s}{v})}{d(t - \frac{s}{v})} \quad (18)$$

$$\text{This is true if } R_o = \sqrt{\frac{L}{C}} \quad (19)$$

The $\sqrt{\frac{L}{C}}$ is called the characteristic resistance of the loss-less line and is designated as R_o or R_c .

Proceeding in a similar manner for ^{the} second terms of e and ^{again} i results in

$$R_o = \sqrt{\frac{L}{C}}.$$

$$\text{Then } i = \frac{e_1}{\sqrt{L/C}} f(t - \frac{s}{v}) - \frac{e_2}{\sqrt{L/C}} f(t + \frac{s}{v}) \quad (20)$$

Now inasmuch as $e_1 f(t - \frac{s}{v})$ is a wave travelling in the positive direction of s and $e_2 f(t + \frac{s}{v})$ is travelling in the negative s direction it is proper to call $e_1 f(t - \frac{s}{v})$ an incident wave and $e_2 f(t + \frac{s}{v})$ a reflected wave.

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Hence

$$e = e_i + e_r \quad \text{where} \quad (21)$$

$$e_i = e_1 f\left(t - \frac{s}{v}\right) \quad \text{and} \quad e_r = e_2 f\left(t + \frac{s}{v}\right)$$

Likewise

$$i = i_i + i_r \quad \text{where} \quad (22)$$

$$i_i = \frac{e_1}{R_o} f\left(t - \frac{s}{v}\right) \quad \text{and} \quad i_r = -\frac{e_2}{R_o} f\left(t + \frac{s}{v}\right)$$

$$\text{where } R_o = \sqrt{L/C}$$

Suppose now a loss-less line is terminated in a resistance R_L , then $e_L = i_L R_L$. Hence

$$e_L = e_{iL} + e_{rL} \quad (23)$$

$$i_L = \frac{e_{iL}}{R_o} - \frac{e_{rL}}{R_o}$$

The solution of these two equations yields

$$\frac{e_{rL}}{e_{iL}} = \frac{R_L - R_o}{R_L + R_o} = K_{eL} \quad (24)$$

$$\frac{i_{rL}}{i_{iL}} = -\frac{R_L - R_o}{R_L + R_o} = K_{iL} = -K_{eL}$$

Equations 24 give the voltage reflection coefficient K_e and current reflection coefficient K_i both at ^{the} load resistance R_L . In other words

$$e_{rL} = K_{eL} e_{iL} \quad \text{and} \quad i_{rL} = K_{iL} i_{iL}$$

For the case in which $R_L = R_0$, $K_{eL} = K_{iL} = 0$ and there is no reflection. That is, all of the energy that reaches the load is dissipated in the load.

When the line is shorted at the load $R_L = 0$, $K_{eL} = -1$ and $K_{iL} = +1$. When the line is open at the load $R_L = \infty$, $K_{eL} = +1$ and $K_{iL} = -1$.

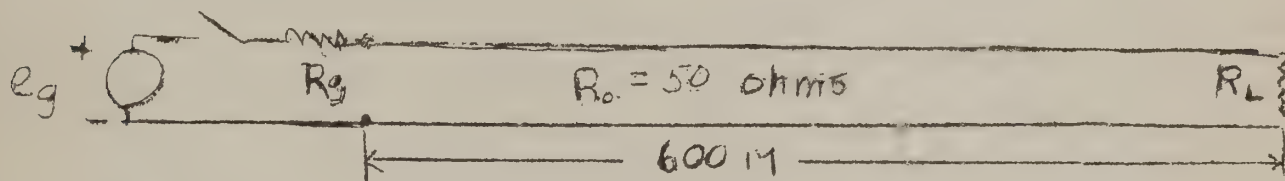


Fig. 6

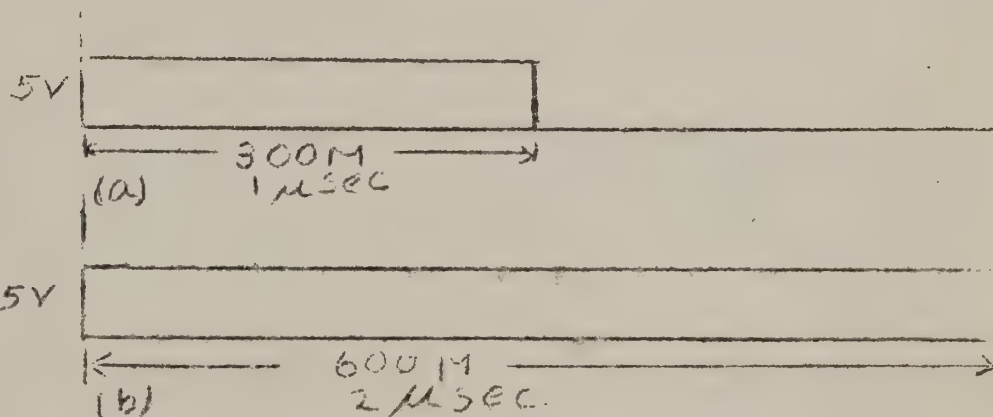


Fig. 7

The source is a 10 volt battery.

$$R_g = 50 \text{ ohms}$$

$$R_L = 50 \text{ ohms}$$

Given a loss-less line, characteristic resistance $R_0 = 50$ ohms, 600 meters long illustrated by Fig. 6. The velocity of propagation is 300 meters per micro second or 3×10^8 meters per second.

Example 1. The source e_g is a 10 volt battery, the resistance $R_g = 50$ ohms and $R_L = 50$ ohms. Figure 7 illustrates the way in which the voltage propagates down the line. When the switch s is closed the resistance to the incident wave is 50 ohms, this is true regardless of the resistance of the load. Hence the line voltage is 5 volts. At the end of 1 μ second the voltage has propagated half way down the line. At the end of 2 μ seconds the entire line is raised to a 5 volt potential. The potential stays at 5 volts as long as the switch remains

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closed. The current becomes $5/50 = .1$ ampere and propagates along with the voltage.

See page 13 for Fig. 8

Example 2. Suppose all conditions are same as in example 1 except $R_L = 150$ ohms. For this condition the voltage reflection coefficient $K_{eL} = \frac{150-50}{150+50} = +.5$ and the current reflection coefficient $K_{iL} = -.5$. Hence the incident voltage and reflected voltage at the load add up to 7.5 volts. Then at the end of 2 μ seconds +2.5 volts propagates toward the source. Since the source resistance $R_g = R_o$ there is no further reflections when the 2.5 volts reaches the source and the line remains at 7.5 volts as long as the switch is closed. The current will reach a steady value of $\frac{7.5}{150} = .05$ amperes. See Fig. 8.

See page 13 for Fig. 9

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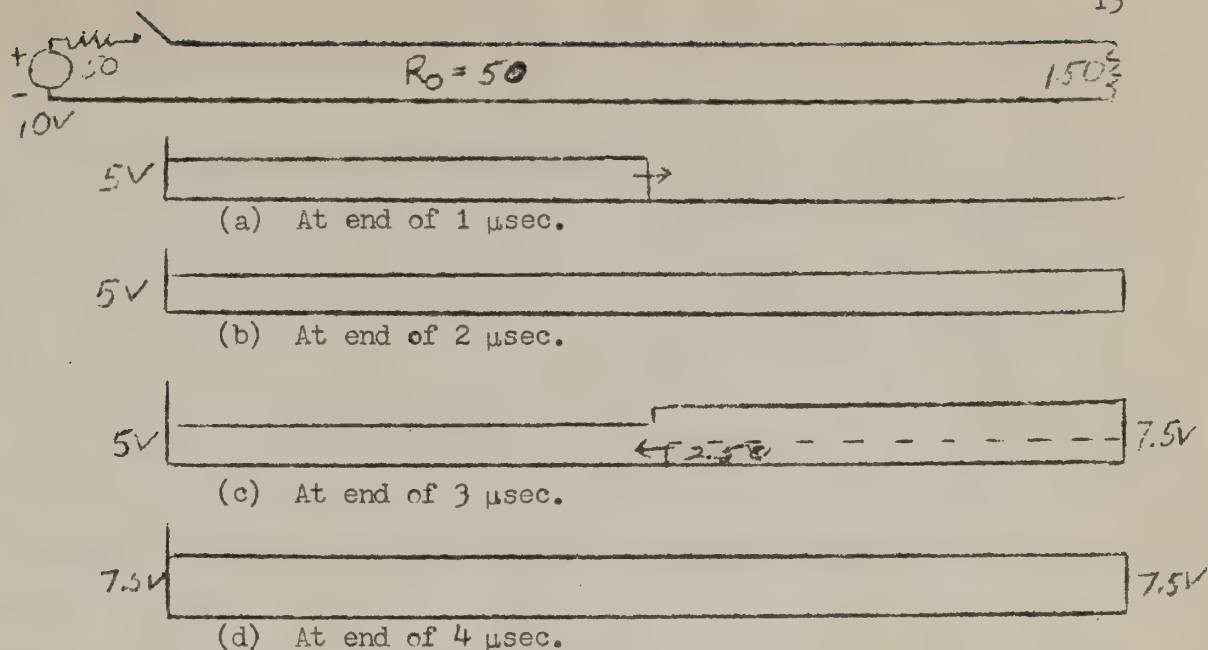


Fig. 8 $e_g = 10V$, $R_g = 50 \text{ ohms}$, $R_L = 150 \text{ ohms}$, for example 2
 $K_{eg} = 0$ and $K_{eL} = +0.5$

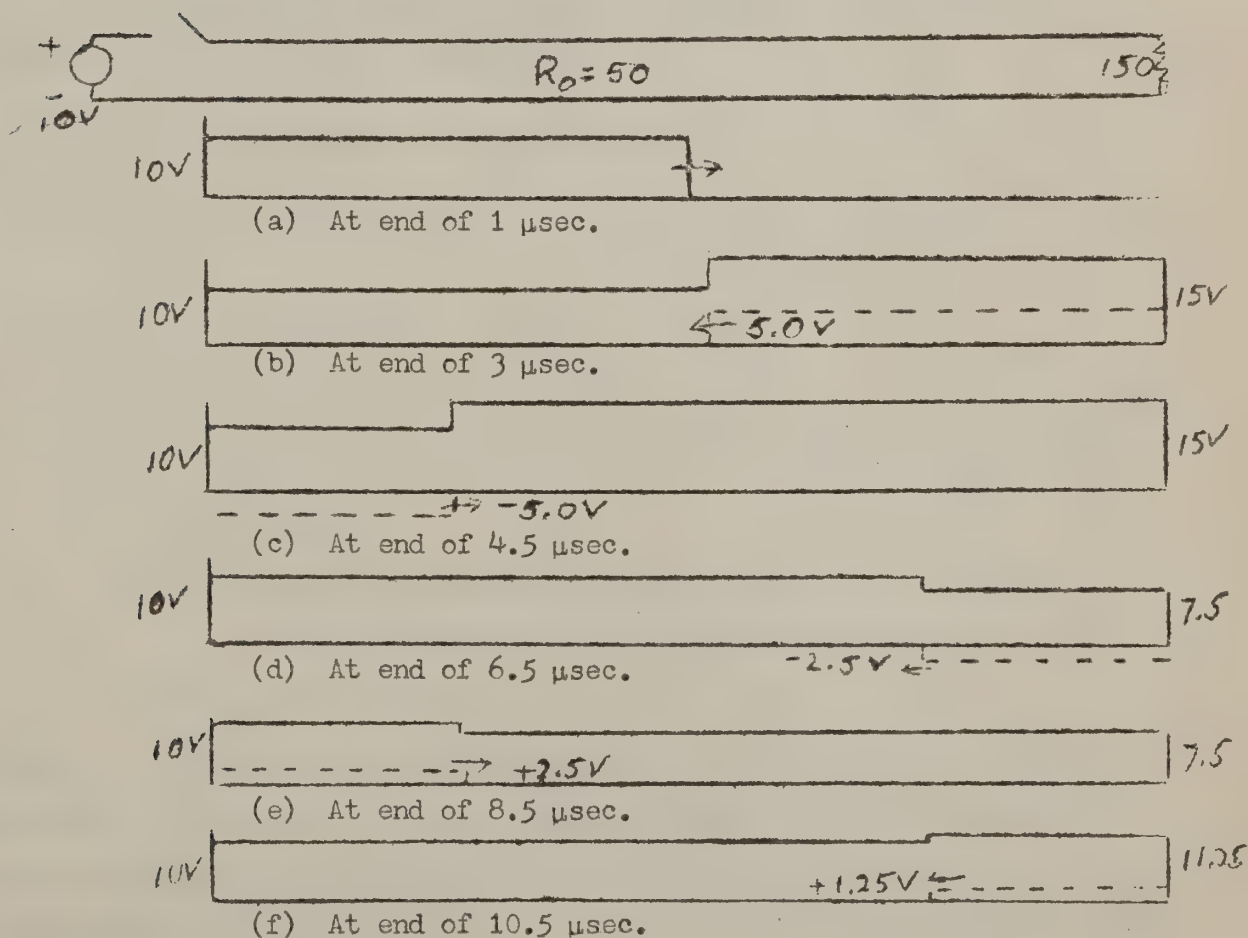


Fig. 9 $e_g = 10V$, $R_g = 0$, $R_L = 150 \text{ ohms}$, for example 3
 $K_{eg} = -1$ and $K_{eL} = +0.5$

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Example 3. Suppose the conditions are the same as in example 2 except $R_g = 0$. In this case the voltage reflection coefficient at the source becomes -1.0 . The load reflection coefficient remains at $+0.5$. Now events are pictured in Fig. 9. Graph (b) shows the first reflection of $+ .5 \times 10 = 5$ volts travelling toward the source. At the source -5 volts are reflected and travel toward the load. Graph (d) shows the second reflection of -2.5 volts travelling toward the source where $+2.5$ volts are reflected. The third reflection results in $+1.25$ volts travelling toward the load. Thus the reflections are dying out and the steady state voltage of the entire line becomes 10 volts.

Example 4. Suppose the conditions are the same as in example 1 except the 10 volt battery (step voltage) is replaced by a 20 volt pulse of $1 \mu\text{sec}$. duration. Since $R_L = R_0$ this pulse which will reach the load in $2 \mu\text{seconds}$, will be dissipated entirely in the load resistor, no reflection.

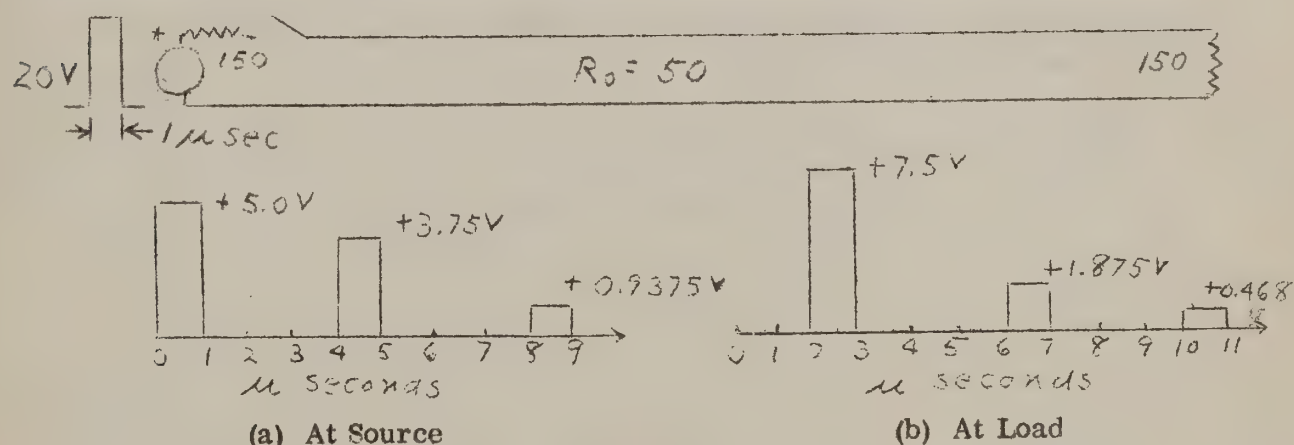


Fig. 10 For each reflection the incident and reflected voltages add to give the total voltage as shown. $K_{eg} = K_{eL} = +.5$ for example 5

Example 5. Suppose the conditions are the same as example 1 except the load resistance is 150 ohms and the source resistance is also 150 ohms. The voltage reflection coefficients at the load and at the source are both equal to 0.5. A $1 \mu\text{sec}$ pulse of 5 volts $[= 50 \times 20 / (150 + 50)]$ travels toward the load and reaches the load in $2 \mu\text{seconds}$. It is reflected at the load as a $+2.5$ volt pulse which travels toward and reaches the source end in another $2 \mu\text{seconds}$. It is reflected at the source as 1.25 volts travels toward the load and reaches the load in an

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additional 2 μsec and so on. Figure 10 represents the total voltage at the source and load for the first several microseconds.

Example 6 Lumped element transmission lines are often used in radar equipment to provide high voltage pulses of short duration for modulation. Figure 11 shows a lumped constant line that is initially charged to 5 KV. Let us determine the load voltage when the switch is closed.

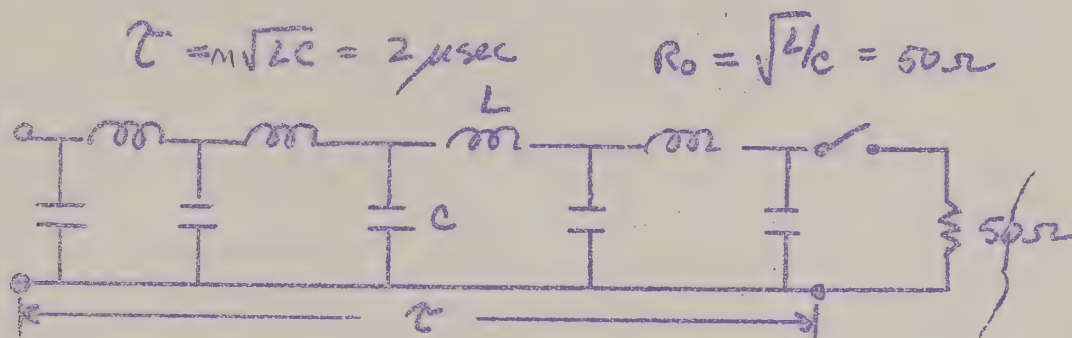


Fig. 11

To determine the initial load voltage at $t = 0+$ we consider the line to exhibit at the load terminals a thevinin source of 5 KV in series with a 50Ω resistance. Hence the initial load voltage is 2.5 KV. The line voltage has thus dropped by 2.5 KV at the load end. This disturbance then propagates down the line toward the open end discharging line capacitance by 2.5 KV as it moves. Upon reflection from the open end ($K = +1$) the -2.5 KV step proceeds back toward the load dropping the line voltage to zero as it goes. Since the line is $\tau = 2 \mu\text{sec}$ long the resultant load voltage is shown in Fig. 12.

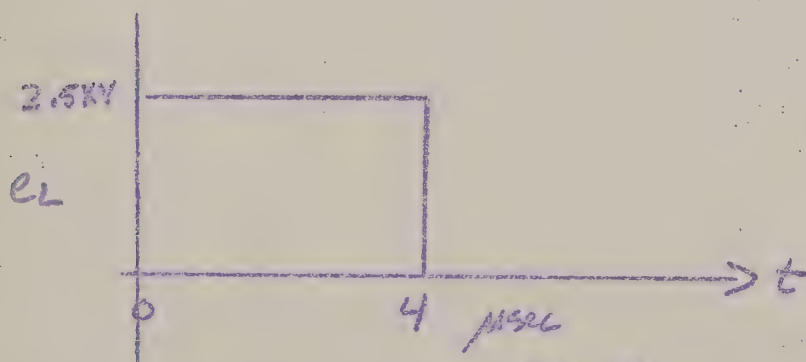


Fig. 12

Example 7

Suppose the system shown in Fig. 13 has reached a steady state condition when at $t = 0$ the switch is opened. Let us determine the load voltage e_L as a function of time.

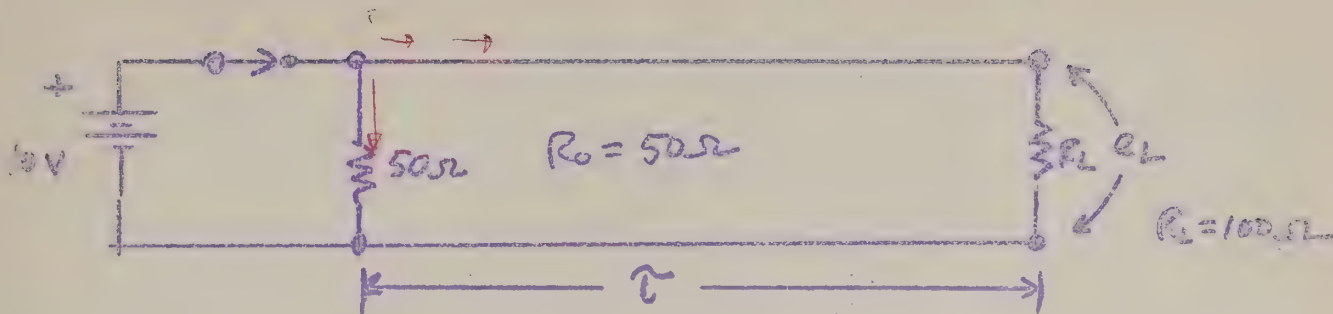


Fig. 13

at $t = 0^-$, $i_{IN} = \frac{10}{50} + \frac{10}{100} = \frac{30}{100}$ a. Hence at $t = 0^+$ we consider the circuit of Fig. 14 wherein a step current of $-\frac{30}{100}$ a

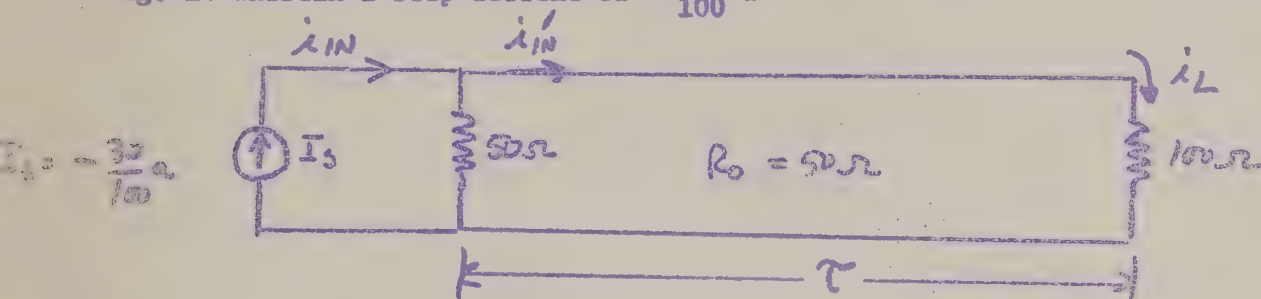


Fig. 14

is applied. Since $R_0 = 50\Omega$ this current divides equally at the input resulting in $i'_{IN} = -\frac{30}{100}$ a. After τ seconds this current reaches the 100Ω load where it sees $R_L = -\frac{1}{3}$. A reflected current step of $+\frac{16}{200}$ a then proceeds toward the generator and where it is absorbed τ seconds later. The resultant load current and voltage are shown in Figs. 15 and 16.

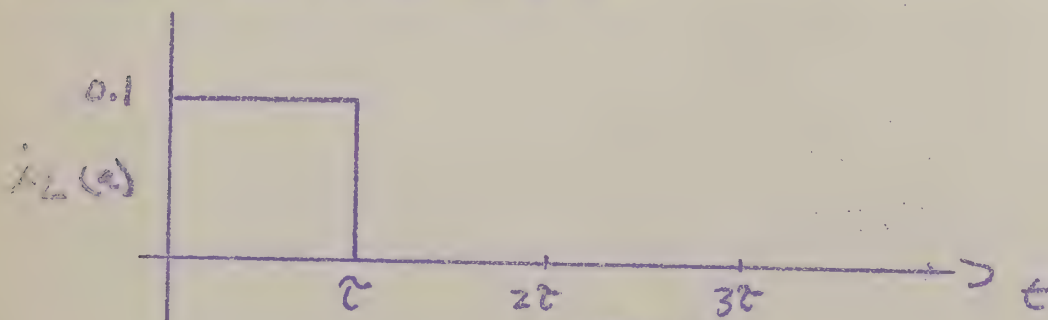
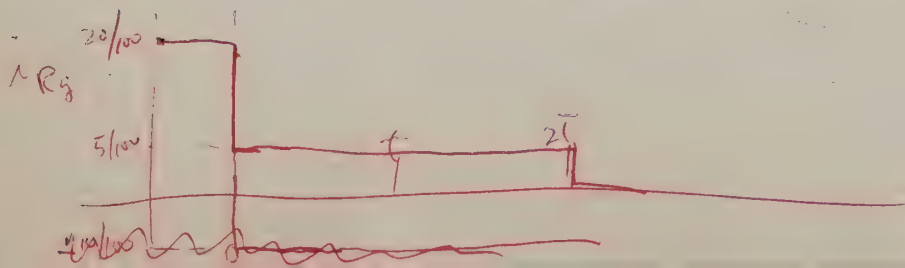


Fig. 15



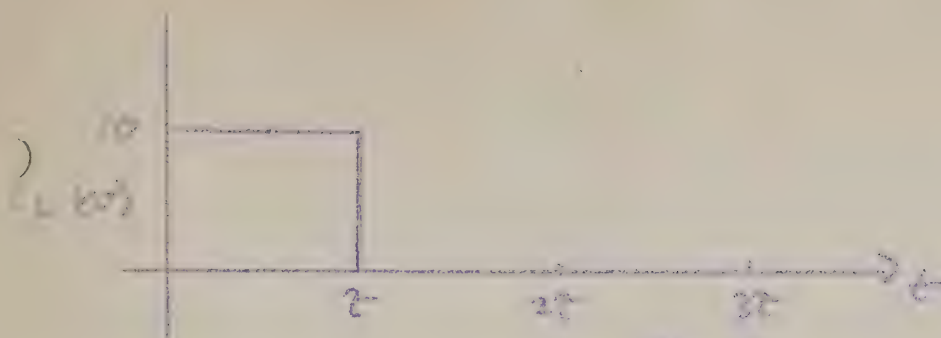


Fig. 16

5. Lines with other than resistive loads

Under transient conditions lines terminated in inductance or capacitance will have reflection coefficients that vary with time. (Note that impedance is a steady state sinusoidal concept and has no application here.) For example, to a voltage step, an inductor looks initially like an open circuit while at $t \rightarrow \infty$ it becomes a short. Hence the voltages and currents on non-resistively terminated lines will vary not only due to the time delay introduced by the line but also due to the finite time required to establish a current in an inductor or a voltage across a capacitor.

Consider the circuit of Fig. 17. Let us determine the load reflection coefficient, load voltage and input voltage for this circuit after the switch is closed.

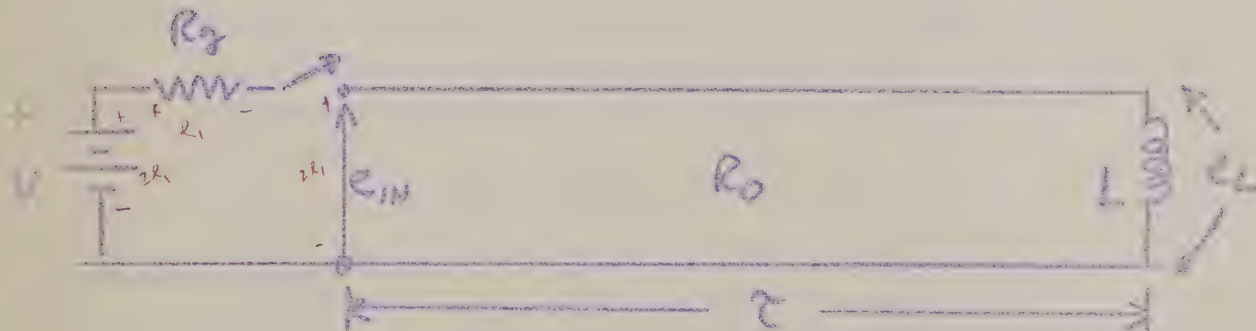


Fig. 17

To find the reflection coefficient we proceed as in the resistive case by expressing the load voltage and current as constrained by the line equations.

At $z = 0$,

$$e_L = e_1 + e_2$$

5

$$i_L = e_1/R_0 - e_2/R_0$$

6

Since the load itself dictates that

$$e_L = L \frac{di_L}{dt}$$

7

Combining Eqs. 5, 6, and 7 results in the differential equation

$$e_1 + e_2 = \frac{L}{R_0} \left(\frac{\partial e_1}{\partial t} - \frac{\partial e_2}{\partial t} \right)$$

8

However, at $t = 0^+$

$$\frac{\partial e_1}{\partial t} = 0$$

9

which simplifies Eq. 8, resulting in

$$\frac{L}{R_0} \frac{\partial e_2}{\partial t} + e_2 + e_1 = 0$$

$$e_1 + e_2 = \frac{L}{R_0} \frac{\partial e_2}{\partial t}$$

With e_1 constant in time. This equation has the solution

$$\ln(e_1 + e_2) = -\frac{L}{R_0} t + C$$

$$e_1 + e_2 = e_0 e^{-R_0 t / L}$$

$$e_1 + e_2 = e_0 e^{-t/\tau}$$

11

It is convenient to measure time from the arrival of the incident wave at the load, i.e., the switch is closed at $t = -\tau$ seconds. Hence, at $t = 0^+$ the load appears as an open circuit and $i_L = 0$

$$e_{2L} = e_{1L}$$

12

Equation 11 then yields

$$e_0 = 2e_1$$

13

Substituting Eq. 13 into Eq. 11 allows the determination of the reflection coefficient at the load as a function of time, viz.

$$\rho_L = \frac{e_2}{e_1} = 2e^{-tR_0/L} - 1$$

14

Figure 18 shows Eq. 14 as well as the load and input voltages as a function of time.

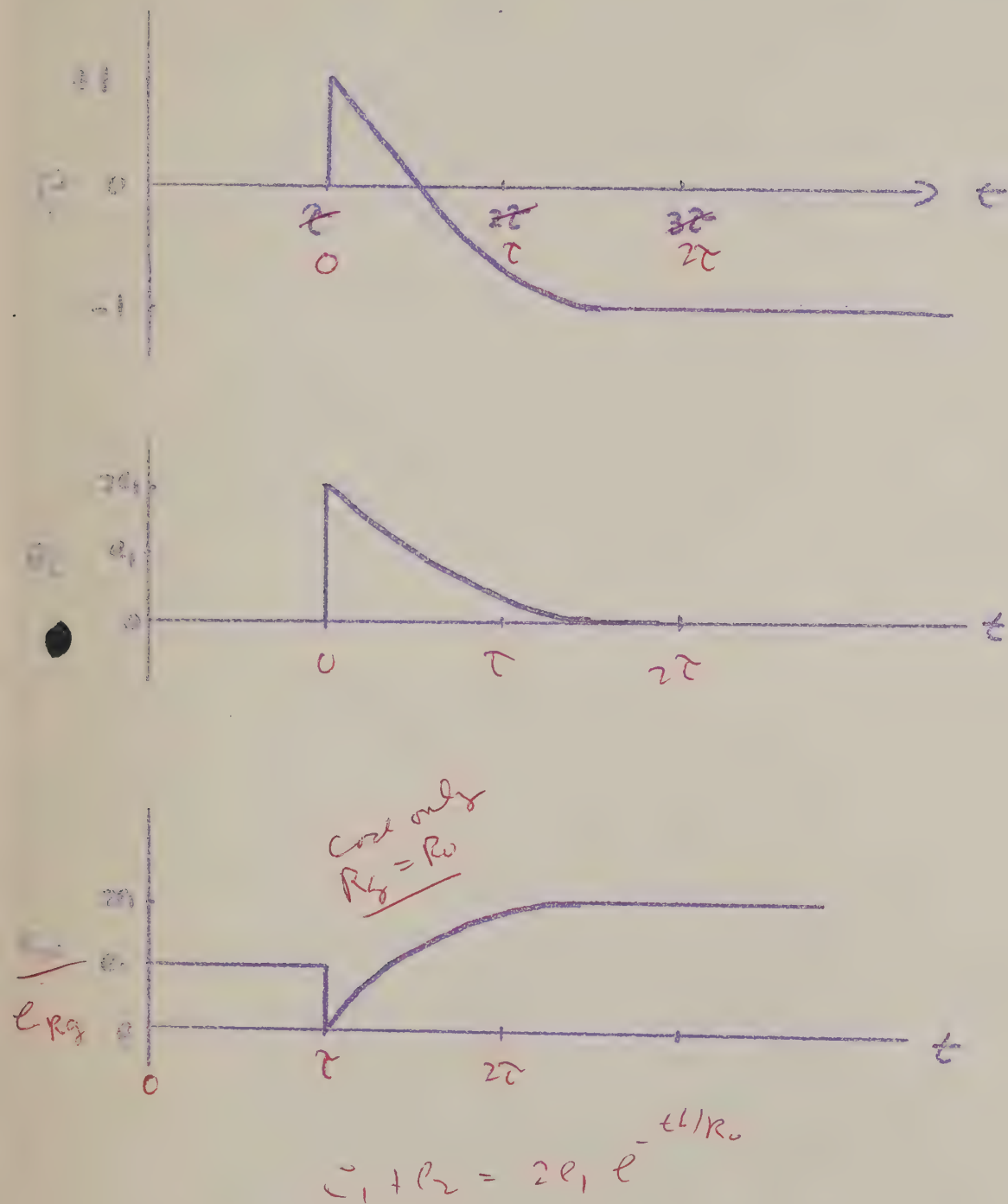
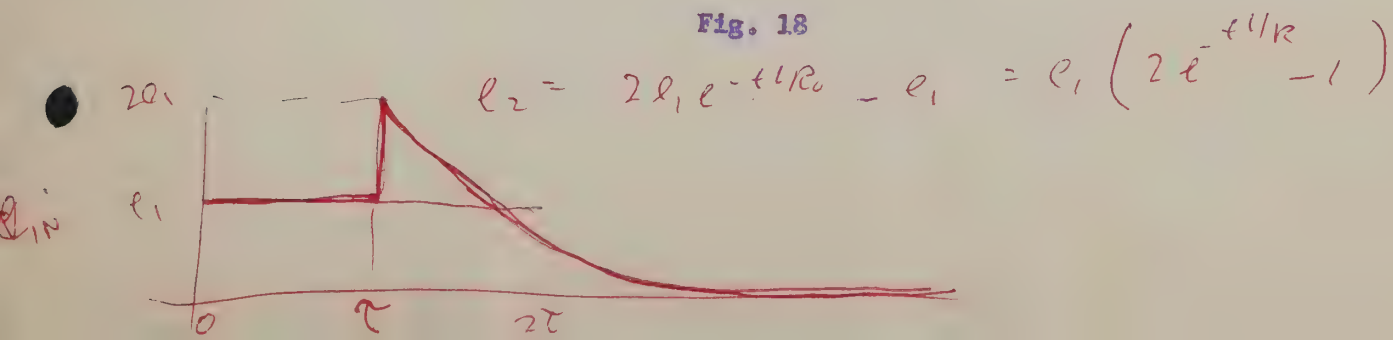
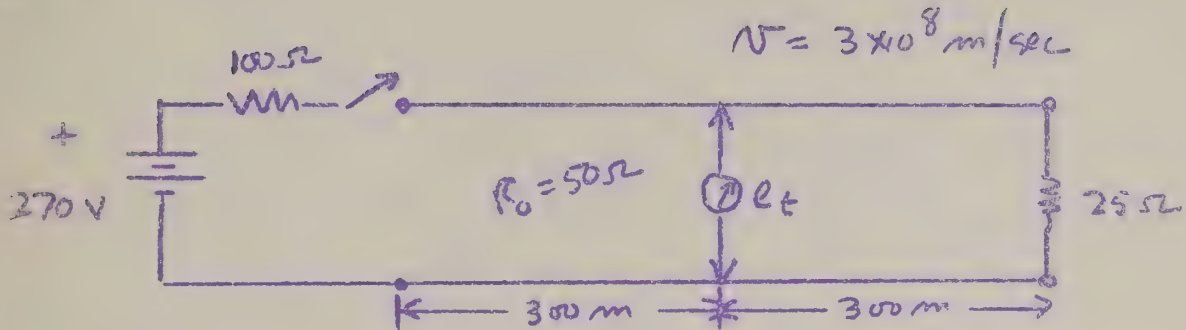


Fig. 18



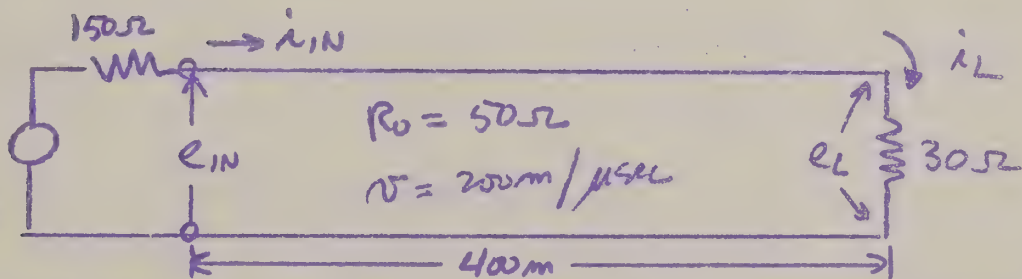
Problems

1)



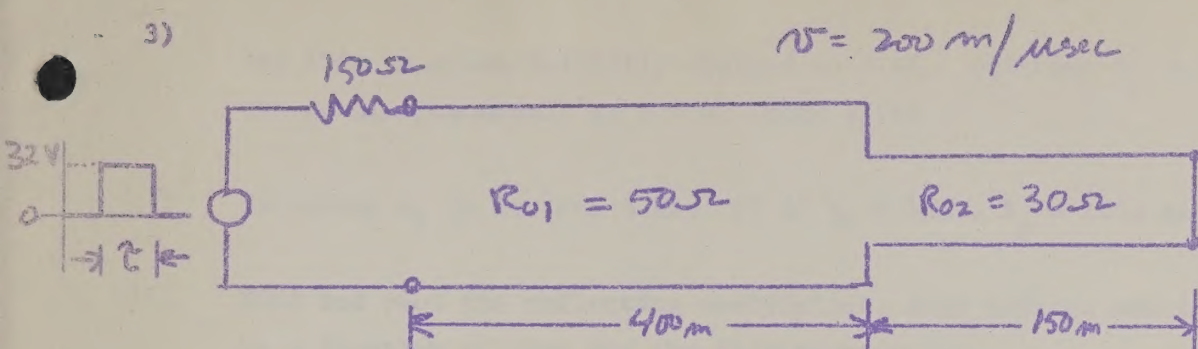
- Determine and plot the voltmeter reading $v(t)$ for the period $t = 0$ to 10 μsec .
- What is the final voltage across R_L ?

2)



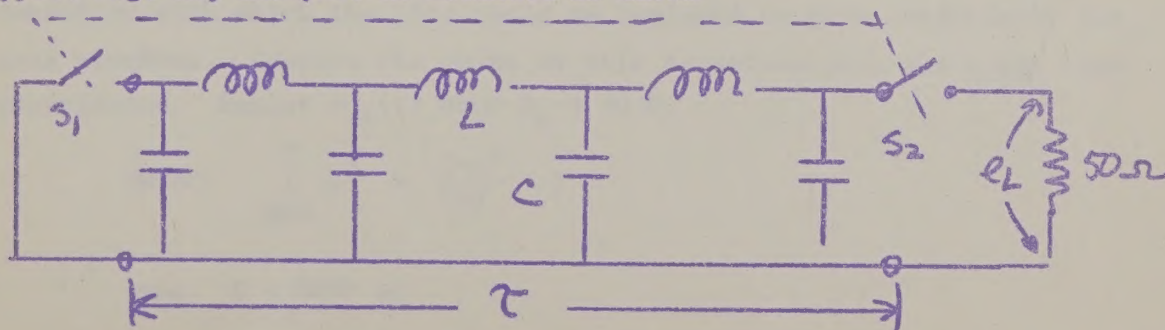
- Determine and plot the input voltage $e_{IN}(t)$ and the input current $i_{IN}(t)$ for the period $t = 0$ to 10 μsec if $\tau = 1 \mu\text{sec}$.
- Repeat (a) for the load voltage $e_L(t)$ and the load current $i_L(t)$.
- Repeat parts (a) and (b) for $\tau = 6 \mu\text{sec}$.
- Determine and plot the voltage and current at the midpoint of the line for $0 \leq t \leq 10 \mu\text{sec}$ for $\tau = 1 \mu\text{sec}$.

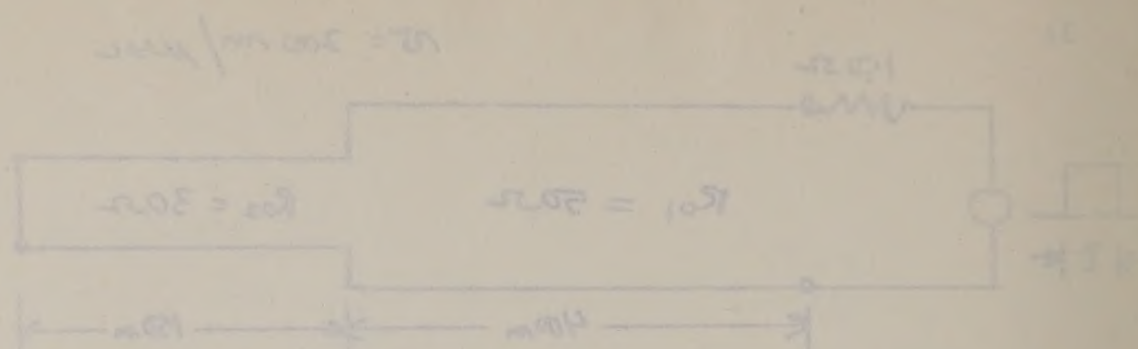
3)



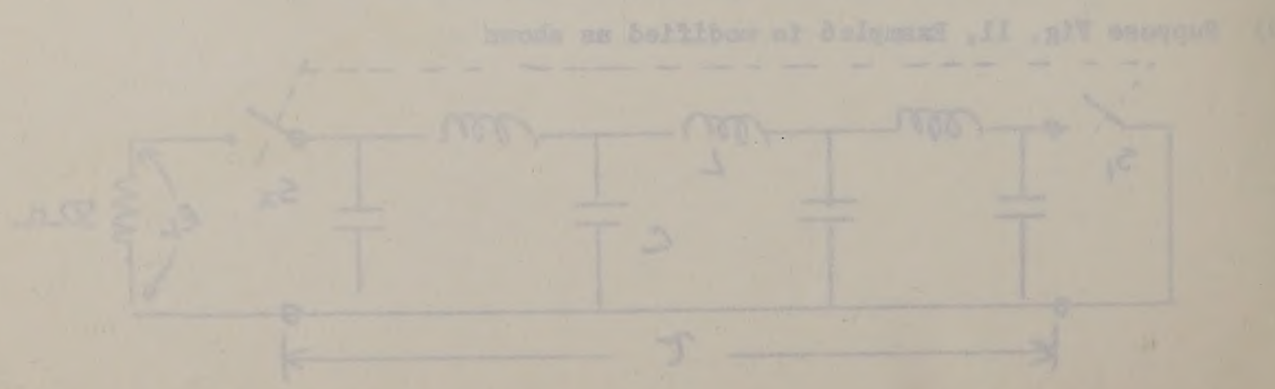
Determine and plot $e_{L_1}(t)$ for the period $0 \leq t \leq 10 \mu\text{sec}$ for $\tau = 1 \mu\text{sec}$.

- 4) Find the voltage as a function of time across the 50Ω resistor in Example 7, Fig. 13.
- 5) Assume in Example 7, Fig. 13 that the switch is open and the line initially uncharged. Find and plot $V_L(t)$ for $t \geq 0$.
- 6) A common dielectric for insulating coaxial lines is polyethylene with $\epsilon_r = 2.25$. Show that $v \approx 200 \text{ m}/\mu\text{sec}$.
- 7) Explain how on oscilloscope, a pulse generator and a precision variable resistor may be used to determine the characteristic resistance of a lossless line.
- 8) A typical coaxial line is designated as RG-8A/U. For this line $R_0 = 50\Omega$ and $v = 200 \text{ m}/\mu\text{sec}$. Find the per meter values of L and C .
- 9) Show in Example 6, Fig. 11 that indeed the delay time $\tau = \sqrt{LC}$ where L and C are the values of the lumped constants. Determine the values of L and C for this example.
- 10) Suppose Fig. 11, Example 6 is modified as shown



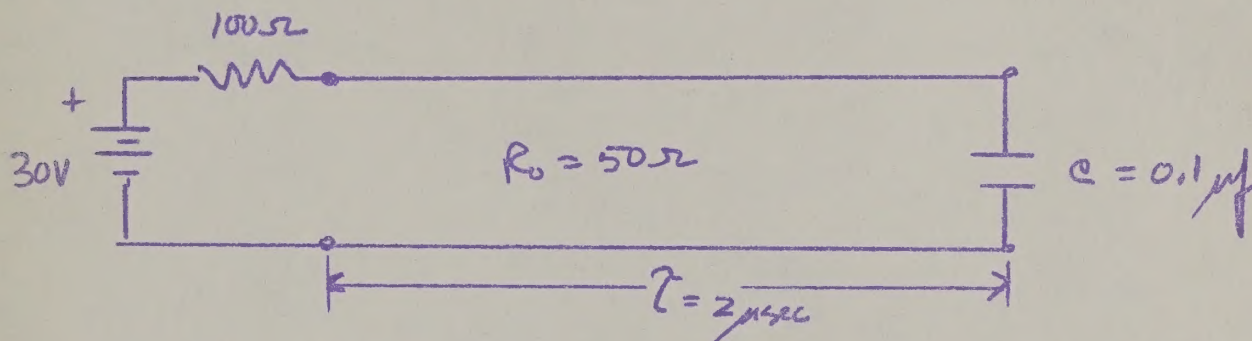


- 1) Determine and plot $v_L(t)$ for the period $0 \leq t \leq 10$ msec for $t = 0$ msec.
- 2) Plot the voltage as a function of time across the 30Ω resistor in $10\mu s$.
- 3) Assume in Example 7, Fig. 11 that the switch is open and the line initially uncharged. Find and plot $v_L(t)$ for $t \geq 0$.
- 4) A common diode for insulating coaxial lines is polyethylene with $\epsilon_r = 2.25$. Show that $v \leq 200$ kV/m.
- 5) Explain how an oscilloscope, a pulse generator and a precision variable capacitor may be used to determine the characteristic impedance of a lossless line.
- 6) A typical coaxial line is designated as RG-58/U. For this line $R_p = 300$ and $v = 200$ m/sec. Find the per meter values of L and C .
- 7) From Example 6, Fig. 11 that indeed the delay time is vL where L and C are the values of the lumped constants. Determine the values of L and C for this example.

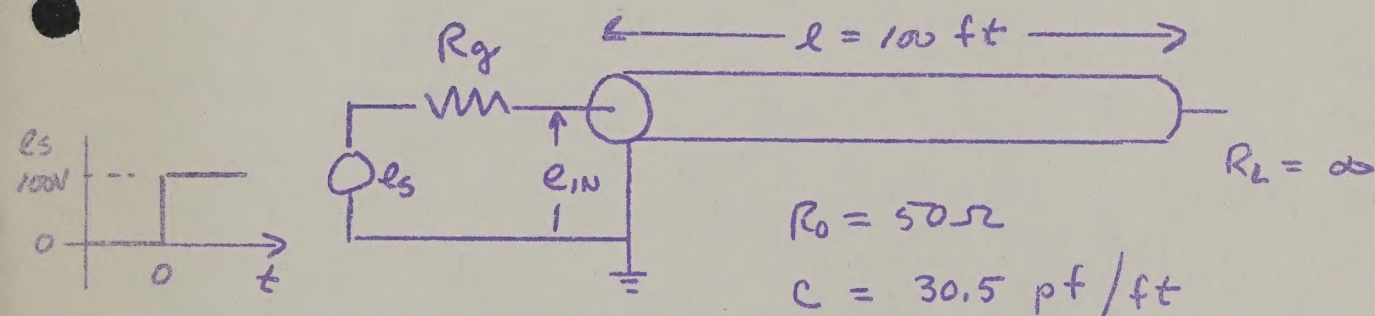


The line is again initially charged to 5 KV. Switches S_1 and S_2 are closed simultaneously at $t = 0$. Find $e_L(t)$

- 11) Determine e_1 in Section 5 Fig. 17 if $R_0 = 50\Omega$, $R_g = 100\Omega$ and $V = 30$ volts.
- 12) Find and plot the reflection coefficient, load voltage and input voltage as a function of time for the line shown below.



13)



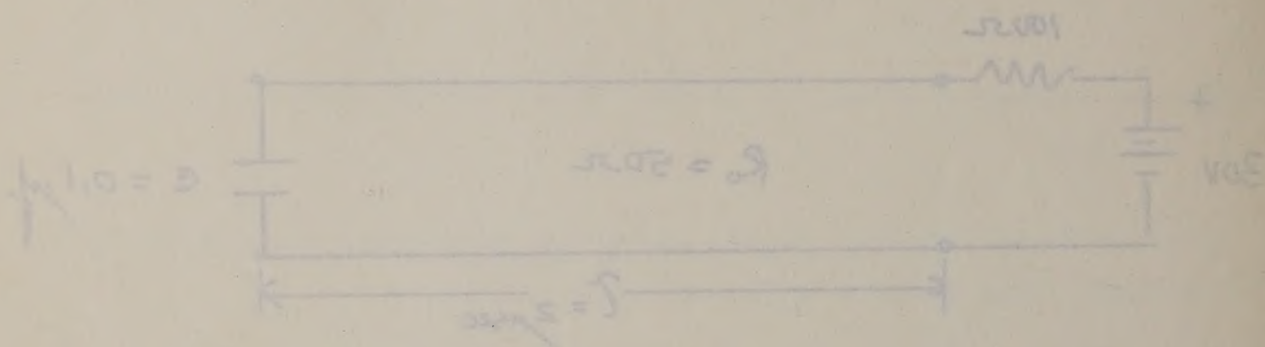
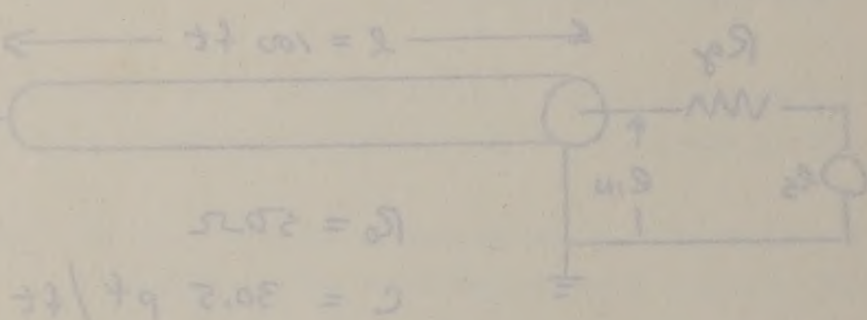
Determine and plot $e_{IN}(t)$ at the input of an open circuit 100 foot length of RG-8A/U coax if $R_g = 1000\Omega$. Find the value of an equivalent capacitor with which the line could be replaced to give essentially the same waveform. Compare the value of this capacitor with the total line capacitance. Replot $e_{IN}(t)$ when $R_g = 50\Omega$.

(Hint: $\sum_{n=0}^{\infty} p^n = \frac{1-p}{1-p}$)

Ans: $C = 2940 \text{ pf}$

$$\Gamma = \frac{Z - Z_0}{Z + Z_0}$$

capacitance. Repeat (c) when $R = 50 \Omega$.
 same waveforms. Compare the value of this capacitor with the total line
 capacitor with which the line could be replaced to give essentially the
 length of RS-800 coax if $R = 1000 \Omega$. Find the value of an equivalent
 inductor and plot Γ at the input of an open circuit 100 feet



as a function of time for the line shown below.
 Find and plot the reflection coefficient, load voltage and input voltage

parameters Γ in Section 2 Fig. 1.12 if $R_s = 50 \Omega$, $R_L = 100 \Omega$ and $V = 30$ volts.

closed at $t = 0$. Find Γ and V at $t = 0$. The line is again initially charged to 30 V. Switches S_1 and S_2 are